ELECTRO-MAGNETISM

$$\oint \boldsymbol{E}\ \boldsymbol{dA} = \frac{\Sigma Q}{\epsilon}$$

$$\oint \boldsymbol{B}\ \boldsymbol{dA} = 0$$

$$\oint \boldsymbol{E}\ \boldsymbol{dl} = -\frac{d}{dt}\int \boldsymbol{B}\ \boldsymbol{dA}$$

$$\oint \boldsymbol{B}\ \boldsymbol{dl} = \mu_0\left(I + \epsilon\frac{d}{dt}\int \boldsymbol{E}\ \boldsymbol{dA}\right)$$

TAMAS ALMOS VAMI

Copyright © 2018 Tamas Almos Vami
MILESTONE INSTITUTE

This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License (CC-BY-NC-ND 4.0)
You are free to copy and redistribute the material in any medium or format under the following terms: (a) you must give appropriate credit (indicating the author's name), provide a link to the license, and indicate if changes were made. You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use; (b) you may not use the material for commercial purposes; (c) if you remix, transform, or build upon the material, you may not distribute the modified material.

This note is a private collection of information by Tamas Almos Vami used for a Electromagnetism module at Milestone Institute, Budapest.

First release, September 2018



Contents

1 Coulomb's law & electric potential 1

1.1 Electric charge 1

1.2 Coulomb's Law 2

1.3 Electric field 3

1.4 Electric potential 4

1.5 Electric field from the potential 5

1.6 Method of image charges 6

1.7 Questions 6

1.8 Exercises 7

2 Gauss's law and its applications 10

2.1 Electric flux 10

2.2 Gauss's law 11

2.3 Applications 11

2.3.1 Electric field of an infinitely large plate 11

2.3.2 Electric field of two infinitely large parallel plates 12

2.4 Exercises 12

3 Capacitance and dielectrics, metals 14

3.1 Capacitors 14

3.2 Parallel-Plate Capacitors 14

3.3 Combinations of Capacitors 15
3.3.1 Parallel combination of capacitors 15
3.3.2 Series combination of capacitors 16
3.4 Energy stored in a capacitor 16
3.5 Effect of dielectrics 17
3.6 Faraday cage 18
3.7 Curvature effects 18
3.8 Questions 18
3.9 Exercises 18

4 Current, resistance and emf 20
4.1 Electric Current 20
4.2 Resistance 21
4.3 Drude model 22
4.4 Exercises 23

5 Direct Current (DC) circuits 25
5.1 Electromotive force 25
5.2 Resistors in Series 26
5.3 Resistors in parallel 26
5.4 Kirchhoff's Rules 27
5.5 RC circuits 27
5.6 Exercises 28

6 Magnetic field & magnetic (Lorentz) force 31
6.1 Brief history 31
6.1.1 Magnetic field 31
6.2 Magnetic force 31
6.3 Particle in a Uniform Magnetic Field 32
6.4 Applications 32
6.5 Magnetic flux 33
6.6 Questions 33
6.7 Exercises 33

7 Induction, Faraday's Law & Lenz's Law 36
7.1 Biot–Savart Law 36
7.2 Magnetic field around a wire 36
7.3 The magnetic force between two parallel conductors 38
7.4 Ampère's Law 38
7.5 Magnetism in Matter 39
7.6 Faraday's (and Lenz's) Law of Induction 39
7.7 Exercises 40

8 EM waves & intro to modern physics 41
8.1 Maxwell's Equations 41
8.2 Differential Maxwell's Equations 42
8.3 EM Waves 42
8.4 Relationship between the magnitudes of B and E in an EM wave 44
8.5 Energy Carried by Electromagnetic Waves 44
8.6 Maximum speed in the universe 44
8.7 Exercises 45

A Natural units 47

B Mathematical background 48
B.1 Functions and derivatives 48
B.2 Vectors 48
B.3 Fields 48
B.4 Vector algebra 49
B.5 Vector analysis 50
B.5.1 A "standard" integral 50
B.5.2 A line integral 50
B.5.3 Stokes' theorem 51
B.5.4 Gauss' theorem 53
B.6 Coordinate systems 54
B.6.1 Polar coordinates 54
B.6.2 Elliptic coordinates 54
B.6.3 Cylindrical coordinates 54
B.6.4 Spherical coordinates 54
B.6.5 Jacobian matrix and determinant 55
B.7 Exercises 55



Preface

This module is concerned with the electromagnetic fields and electromagnetic radiation that pervade the world around us. In this module we will cover the basic principles, including the electric field and potential, Coulomb's law, Gauss's law, currents, the magnetic field, the Lorentz force, Faraday's law, or Lenz's law, and we will finish with electromagnetic waves and an introduction to modern physics.

In this book, I'll cover most of Electromagnetism concentrated in 8 modules.

This book is heavily under construction. In case you find any errors in it please send an email to

vami@milestone-institute.org

Until so far here is a Feynman diagram showing that electrons can emit light in the environment of some nuclei.

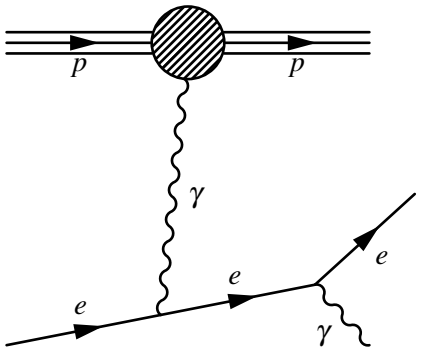
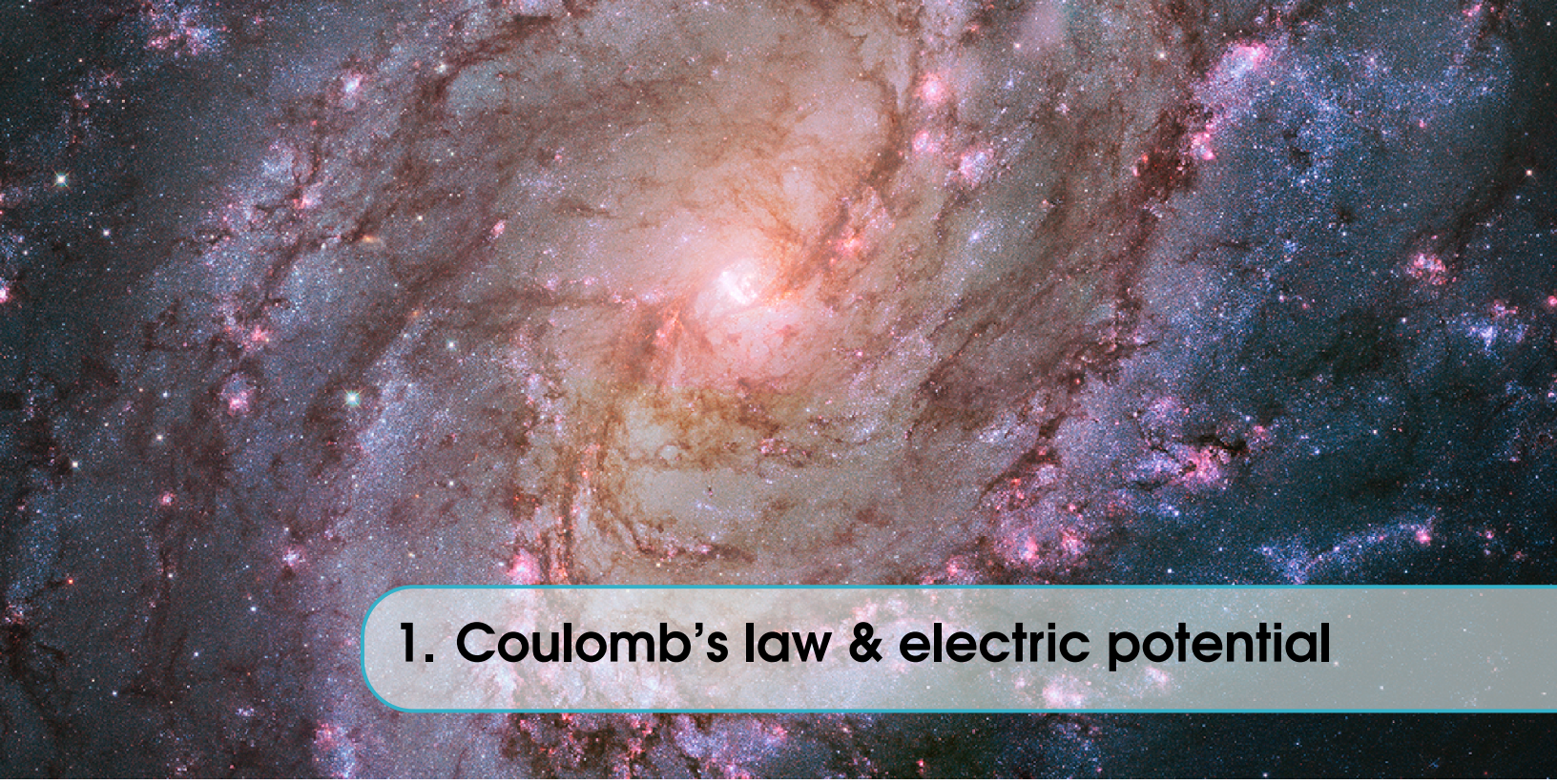


Figure 1: Feynman diagram for Bremsstrahlung



1. Coulomb's law & electric potential

1.1 Electric charge

Long history (from the Greeks), Modern description is Quantum Electrodynamics (R.P. Feynman). One of the source of the charge is a particle called electron (e^-)

Charge is conserved

The algebraic sum of all electric charges in any closed system is constant.

As a consequence of this when you rub a rod with a fur some electrons move from the fur to the rod. But since the sum must be constant, the rub will be more negative then the fur. The rob is "negatively charged" and the fur is "positively charged".

From observation (experiments) we know the following law:

Forces between charges

Two positive charges or two negative charges repel each other. A positive charge and a negative charge attract each other.

What is the unit of the electric charge? Every observable amount of electric charge is always an integer multiple of this basic unit.

Unit of charge

The natural unit of charge is the magnitude of the charge of the electron or the proton. Having $6.25 \cdot 10^{18}\, e^-$ will lead to 1 Coulomb (C):

$$1\,C = 6.25 \cdot 10^{18}\, e^-$$

One other source of charge is induction. Let us have a metal ball on an insulating sand on the ground. Originally the ball is uncharged. Then let us approach the ball with a negatively charged rod. The charges on the rod will repel the electrons, and they will create zones of negatively and

positively induced charge (closer to the rod it will be positive, further it will be negative). As a next step let us connect the ball to the ground with a wire, that will lead the electrons in the negative zone of the ball into the ground. After removing the wire the ball will have an overall electron deficiency (i.e. it will have a "positive" charge).

Materials can be characterized by how much they conduct electricity. We have the situation when all electrons are bound to atoms and cannot move freely through the material. These are called insulators. Then if some of the electrons are free electrons that are not bound to atoms and can move relatively freely through the material, they are called conductors.

1.2 Coulomb's Law

Coulomb's Law

The magnitude of the electric force between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

$$F = k\frac{q_1 q_2}{r^2}$$

where k is a constant depending on the units used, in SI it is

$$k = 8.988 \cdot 10^9 \frac{Nm^2}{C^2}$$

or

$$k = \frac{1}{4\pi\varepsilon_0}$$

where ε_0 is vacuum permittivity.

A side note – the fine-structure constant

Checking this interaction between two electrons leads to very deep understanding of Nature. In the units of $\hbar = c = \varepsilon_0 = 0$ we have

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.035} \approx \frac{1}{137}$$

that we call the the fine-structure constant. This is a dimensionless number so by releasing the unit choice above we can have it as

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$$

Introducing a bit of generalization: let us denote the coordinate of one of the charges with a vector: $\mathbf{x_1}$ and the second charge with the vector: $\mathbf{x_2}$. Then the force will only depend on the relative distance of the two charges, i.e. on $\mathbf{x_2} - \mathbf{x_1}$. This is a vector and depending on the sign of the charges is shows the direction of the force that will act on the to charges.

We cannot divide with a vector so we have to use a trick, that is

$$\mathbf{a} = \frac{\mathbf{a}^2}{|\mathbf{a}|}$$

With this the Coulomb's Law in its most general form:

Coulomb's Law

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{(\mathbf{x}_2 - \mathbf{x}_1)}{(|\mathbf{x}_2 - \mathbf{x}_1|)^3} \tag{1.1}$$

where ε_0 is still vacuum permittivity.

1.3 Electric field

It has been a longstanding question that when two particle in empty space interact, how does each one know that the other is there. What goes in the space between them to communicate the effect of each on to the other? Faraday's idea was that the charge creates an electric field (denoted with $\mathbf{E}$) that is understood everywhere in space, and then the other charge interacts with this field and this interaction is what we call the force between them.

Electric field

The electric force on a charged body is exerted by the electric field created by the other charged bodies.

Definition 1.3.1 — Electric field. Let us define a test charge (q_0). If the test charge experiences an electric force then there is an electric field at the point:

$$\mathbf{E} = \frac{\mathbf{F}}{q_0}$$

Corollary 1.3.1 The unit of the electric field is

$$[\mathbf{E}] = \frac{N}{C}$$

Based on Equation 1.1 and the definition 1.3.1 the general way to calculate the electric field in any give position ($\mathbf{x}$) is going to be

Electric field

$$\mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\varepsilon_0} \frac{(\mathbf{x} - \mathbf{x}_1)}{(|\mathbf{x} - \mathbf{x}_1|)^3} \tag{1.2}$$

where ε_0 is still vacuum permittivity.

Here we replaced $\mathbf{x_2}$ with $\mathbf{x}$ and we set q_2 to be q_0, $\mathbf{x_1}$ denotes the position of the charge that creates the electric field.

Some things to note here: the electric field is a vector field and its direction is radial (given one charge that creates the field). The interaction is done via a particle called photon, that is considered to be a massless particle.

The electric field is best described with the so called electric field lines. They point away from positive charges and towards the negative charges. At each point in space the electric field vector is tangent to the field line in that given point. The density of the field lines is increasing when the field is strong.

We can generalize the expression in equation 1.2 to the situation when we have more then one charge. Then the sum of the charges will matter that we will denote with

$$q_{sum} = \sum_i q_i$$

where the i indexes the charges. With this we have

Electric field

$$\mathbf{E}(\mathbf{x}) = \sum_i \frac{q_i}{4\pi\varepsilon_0} \frac{(\mathbf{x}-\mathbf{x}_i)}{(|\mathbf{x}-\mathbf{x}_i|)^3} \tag{1.3}$$

where ε_0 is still vacuum permittivity and $\mathbf{x}_i$ is the position of the i-th charge.

What if we do not have a discrete number of charge but rather a continuous function? Then the summation is replaced by an integral, the q_i-s are replaced with the infinitesimal dq.

Electric field

$$\mathbf{E}(\mathbf{x}) = \int dq \frac{1}{4\pi\varepsilon_0} \frac{(\mathbf{x}-\mathbf{x}_0)}{(|\mathbf{x}-\mathbf{x}_0|)^3} \tag{1.4}$$

where ε_0 is still vacuum permittivity and $\mathbf{x}_0$ is the position parameter of the continuous charge.

Definition 1.3.2 — Charge density functions. If the charge is continuously distributed in a certain material, it is helpful to introduce the charge density functions. At this point it is usual to distinguish different dimensional cases:

- 1D: $dq = \lambda(\mathbf{x}_0)\, dx$
- 2D: $dq = \sigma(\mathbf{x}_0)\, dA$
- 2D: $dq = \rho(\mathbf{x}_0)\, dV$

where λ, σ, ρ are charge density functions.

For example a wire can be considered a 1D object as its diameter is negligible to its length. Then one could measure that there are 10 electrons in every nanometer, so its density $\lambda = 10\frac{1}{nm}$, having 1 meter long cable one can calculate the charge of this 1D object.

Please note that this number does not have to be constant, it could depend on where we are looking in the object.

1.4 Electric potential

Let us identify the charge and the field as a system. If the charge is free to move, it will do so in response to the electric force. Therefore, the electric field will be doing work on the charge. This work is internal to the system

Electric work

$$W_{int} = \mathbf{F} \cdot \mathbf{dx} = q\mathbf{E} \cdot \mathbf{dx} \tag{1.5}$$

where **dx** is the vector describing a little path the charge takes ($\mathbf{dx} = (dx, dy, dz)$).

Internal work done in a system is equal to the negative of the change in the potential energy of the system.

Potential energy of the system

$$dU = -W_{int} = -q\mathbf{E} \cdot \mathbf{dx} \tag{1.6}$$

$$\Delta U = \int dU = -q \int_A^B \mathbf{E} \cdot \mathbf{dx} \tag{1.7}$$

i.e. the charge times the line integral through the path of the charge in the electric field.

The force is conservative, this line integral does not depend on the path taken from A to B.

Dividing the potential energy by the charge gives a physical quantity. It depends only on the source charge distribution and it has a value at every point in an electric field. This quantity is called the **electric potential** (or simply the **potential**):

Electric potential

$$V = \frac{U}{q} \tag{1.8}$$

The **potential difference** between two points A and B in an electric field is defined as the change in electric potential energy of the system when a charge q is moved between the points

Electric potential

$$\Delta V = \frac{\Delta U}{q} = -\int_A^B \mathbf{E} \cdot \mathbf{dx} \tag{1.9}$$

Because electric potential is a measure of potential energy per unit charge, the SI unit of both electric potential and potential difference is joules per coulomb, which is defined as a **volt** (V):

$$1V = 1J/C$$

1.5 Electric field from the potential

Let us say that the potential is given. How do you extract the electrif field out of that? Let us take 1D firt, then the equation

$$dV = -\mathbf{E} \cdot \mathbf{dx}$$

will not be a vector anymore but it will be

$$dV = -E_x \cdot dx$$

from this we would naively say that the electric field in the x direction is going to be

$$E_x = -\frac{dV}{dx} \tag{1.10}$$

in which we recognize one way of denoting a derivative. As it is mentioned in the Appendix the 3D version of this is going to be

Electric field from the potential

$$\mathbf{E} = -\nabla V = -grad(V) \tag{1.11}$$

which essentially means componentwise the following:

$$E_x = -\frac{\partial V}{\partial x} \quad E_y = -\frac{\partial V}{\partial y} \quad E_z = -\frac{\partial V}{\partial z}$$

Please note that with this we can have a formulate for the potential that is similar to what we saw earlier with the electric field.

Electric potential produced by charges

$$V(\mathbf{x}) = \int \frac{1}{4\pi\varepsilon_0} \frac{1}{|\mathbf{x}-\mathbf{x}_0|} dq \tag{1.12}$$

So essentially by finding the electric potential functions of the charges, then applying the superposition principle (i.e. the potentials can be added) we can derive everything.

1.6 Method of image charges

Consider a grounded conducting plane next a set of charges. The electric field due to charge induced on the plane is the same as if a charge of opposite sign is behind the plane the same distance the original charge is from the plane. This idea is visualized on Figure 1.1.

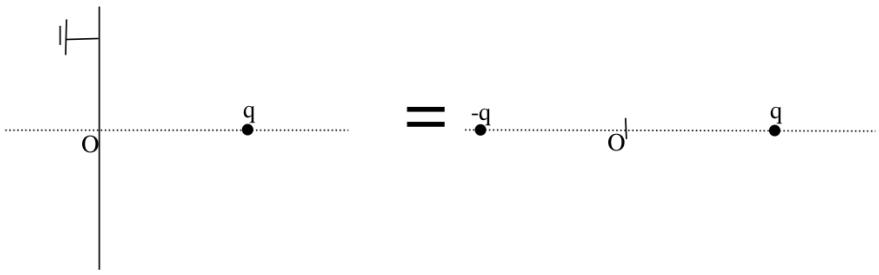


Figure 1.1: The Method of images for a single point charge.

1.7 Questions

- Why is the charge conserved? Is that really always the case? Research "CP violation" for interesting phenomena in this topic.

- Why is the magnitude of the charge of the electron the unit? Are there particles with fractional charge?
- Research what is the Milliken experiment.
- How do we know what is the mass of the photon? Its mass is actually measured to be

$$m_\gamma < 1.6 \cdot 10^{-47} g$$

1.8 Exercises

Exercise 1.1 — 2 point. The electron and the proton of a hydrogen atom are separated (on the average) by a distance of approximately $5.3 \cdot 10^{-11}\, m$. (constants: $k = 9 \cdot 10^9\, N \cdot m^2/C^2$ and $G = 6.67 \cdot 10^{-11}\, N \cdot m^2/kg^2$)

(a) Find the magnitudes of the electric force and the gravitational force between the two particles

(b) and compare the ratio of the two.

■

Exercise 1.2 — 2 point. Consider three point charges located at the corners of a triangle as shown in Figure 1.2, where $q_1 = q_3 = 5.00\,\mu C$, $q_2 = -2.00\,\mu C$ and $a = 0.100\,m$.

(a) Find the resultant force exerted on q_3.

■

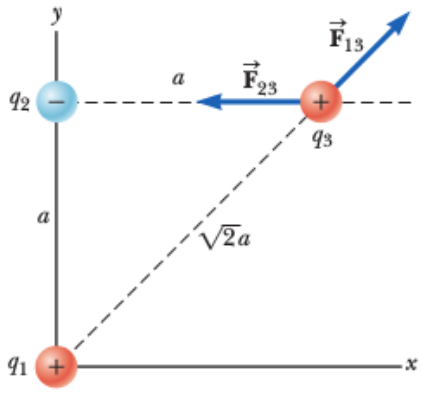


Figure 1.2: The force exerted by q_1 on q_3 is F_{13}. The force exerted by q_2 on q_3 is F_{23}. The resultant force F_3 exerted on q_3 is the vector sum of F_{13} + F_{13}.

Exercise 1.3 — 3 point. A ring of radius a carries a uniformly distributed positive total charge Q.

(a) Calculate the electric field due to the ring at a point P lying a distance x from its center along the central axis perpendicular to the plane of the ring (Figure 1.3).

■

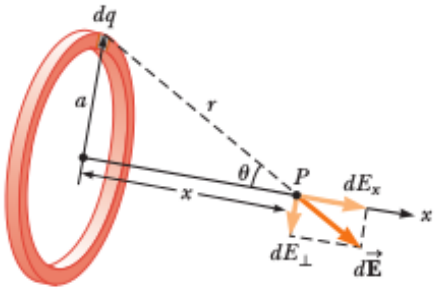


Figure 1.3: A uniformly charged ring of radius a. The field at P on the x axis due to an element of charge dq.

Exercise 1.4 — 3 point. Calculate the electric potential in the following situation: there is a source of charge that creates an electric field like $\mathbf{E} = r^2\mathbf{e}_r$, where $\mathbf{e}_r$ is the radial unit vector. Now let us assume that a particle is coming on a half cyclic path r_0 distance from that point as visible in Figure 1.4. What is the potential difference on this (dark blue) path between A and B? ■

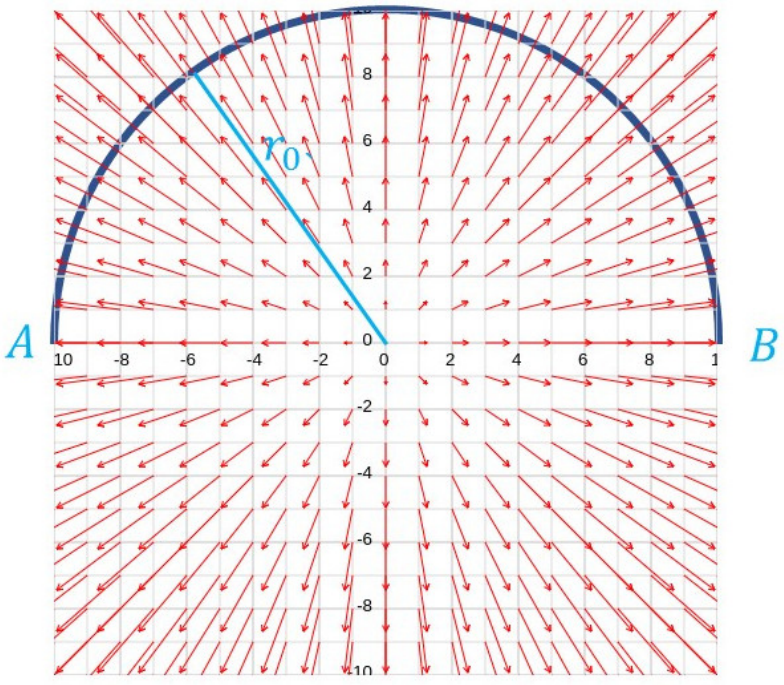


Figure 1.4: A special situation for an electric potential calculation.

Exercise 1.5 — 3 point. The potential outside of a charged conducting cylinder with radius R

and charge per unit length l is given by the below equation.

$$V = \frac{l}{2\pi\varepsilon_0} ln\left(\frac{R}{r}\right) \tag{1.13}$$

What is the electric field at a point located at a distance r from the surface of the cylinder? ■

Exercise 1.6 — 3 point. A nonuniformly charged ring of radius R carries a linear charge density of $\lambda(\theta) = bsin^2\theta$. Calculate the potential at the center of the ring. What is the electric field there? ■

Exercise 1.7 — 5 point. A nonuniformly charged hemispherical shell of radius R has surface charge density $\sigma(\theta) = bcos^3\theta$. Calculate the potential at the center of the opening of the hemisphere (the origin). What is the electric field there? ■

Exercise 1.8 — 5 point. An insulator ball with radius r and charge Q is hang above salty water. The distance between the bottom of the ball and the water surface is r again. Due to the force between the charge and the ions in the water there will be a bump in the water surface. What is the height of this bump? ■



2. Gauss's law and its applications

Given any general distribution of charge, we surround it with an imaginary surface that encloses the charge. We look at the electric field at various points on the surface then Gauss's law is a relationship between the field at all the points on the surface and the total charge enclosed within the surface.

2.1 Electric flux

First, we need to introduce the definition of flux.

Definition 2.1.1 — Electric flux. Flux is the product of the magnitude of the electric field and surface area perpendicular to the field.

Let us take the electric field perpendicular to the area, then the electric flux Φ_E will be

$$\Phi_E = E_\perp A = E cos\theta A = \mathbf{E}\cdot\mathbf{A} \tag{2.1}$$

where $\mathbf{A}$ is a vector perpendicular to the area and its magnitude is the surface area.

The unit is given by

$$[\Phi_E] = N\frac{m^2}{C}$$

What happens is the electric field varies from point to point over the area? What if A is part of a curved surface? We calculate the electric flux through each element, then we integrate the results to obtain the total flux.

$$\Phi_E = \sum E_i dA_i \tag{2.2}$$

which in the continuous case will become

$$\Phi_E = \int E cos\theta dA = \int \mathbf{E}\cdot\mathbf{dA} \tag{2.3}$$

Flux

So in the end the flux is calculated as the surface integral of the electric field:

$$\Phi_E = \int \mathbf{E} \cdot \mathbf{dA} \tag{2.4}$$

In case the surface is closed, we call it a closed surface integral, denoted as

$$\Phi_E = \oint \mathbf{E} \cdot \mathbf{dA}$$

2.2 Gauss's law

Next step is to calculate what is the flux in a sphere?

$$\Phi_E = \mathbf{E} \cdot \mathbf{A} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \cdot (4\pi R^2) = \frac{q}{\varepsilon_0} \tag{2.5}$$

Notice that the flux is independent of radius R of the sphere!

What is the flux in a non-spherical surface? Well, it can be shown that the results will be equivalent.

What if we have more charges? Then q is replaced with the sum of the charges that are **inside** of the surface under investigation:

$$\Phi_E = \frac{\sum q_{inside}}{\varepsilon_0} \tag{2.6}$$

Gauss's law

The total flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ε_0

$$\Phi_E = \int \mathbf{E} \cdot \mathbf{dA} = \frac{\sum q_{inside}}{\varepsilon_0} \tag{2.7}$$

There is one more thing that we can do, i.e. having a continuous charge distribution as defined in Definition 1.3.2. In this case we will use the 3D case and denote the charge density function with ρ.

Gauss's law for continuous charge distribution

The total flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ε_0

$$\Phi_E = \int \mathbf{E} \cdot \mathbf{dA} = \frac{1}{\varepsilon_0}\int \rho dV \tag{2.8}$$

2.3 Applications

2.3.1 Electric field of an infinitely large plate

Let us have an infinity large plane with $\sigma = \frac{Q}{A}$ charge density uniformly distributed on it. What is going to be the electric field created by this?

Let us choose the Gauss surface to be a cylindrical surface whose top/bottom is $\pm d$ distance from the surface. The walls of the cylinder are perpendicular and the planes of it are parallel to the surface under study. Based on symmetry principles the electric fields cancel each other on the perpendicular walls, thus we only have a contribution from the parallel planes.

$$\int EdA = E(d)(2r^2\pi) + E(-d)(2r^2\pi) = 2E(2r^2\pi) \tag{2.9}$$

As due to symmetry principles again we have $E(d) = E(-d)$.

The right-hand-side of the Gauss law will be

$$\frac{Q_{inside}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} = \frac{\sigma(2r^2\pi)}{\varepsilon_0} \tag{2.10}$$

Putting this together

$$2E(2r^2\pi) = \frac{\sigma(2r^2\pi)}{\varepsilon_0} \tag{2.11}$$

which leads to

$$E = \frac{\sigma}{2\varepsilon_0} \tag{2.12}$$

Please note that the results do not depend on the distance!

2.3.2 Electric field of two infinitely large parallel plates

Let us have the two plates be d distance away from each-other, and the one above have σ charge density while the one in the bottom a $-\sigma$ charge density. As we saw above one plate creates a $E = \frac{\sigma}{2\varepsilon_0}$ electric field. We have three regions in our situation now: (a) above the top plate, (b) between the top and the bottom plate, (c) below the bottom plate. In the case of (a) the top plate creates an $E = \frac{\sigma}{2\varepsilon_0}$ pointing outwards while the bottom plate creates n $E = \frac{\sigma}{2\varepsilon_0}$ pointing inwards, thus they cancel each-other. The opposite of the situation happens in the (c) region. Only the (b) region give a contribution where the outward pointing electric field of the top plate is parallel to the inward pointing electric field of the bottom plate. By adding the two we get that the electric field is

$$E = \frac{\sigma}{\varepsilon_0} \text{ if we are between to two plates, otherwise zero} \tag{2.13}$$

2.4 Exercises

Exercise 2.1 — 4 point. An insulating solid sphere of radius a has a uniform volume charge density r and carries a total positive charge Q.

(a) Calculate the magnitude of the electric field at a point outside the sphere. (Figure 2.1.a)

(b) Find the magnitude of the electric field at a point inside the sphere (Figure 2.1.b).

■

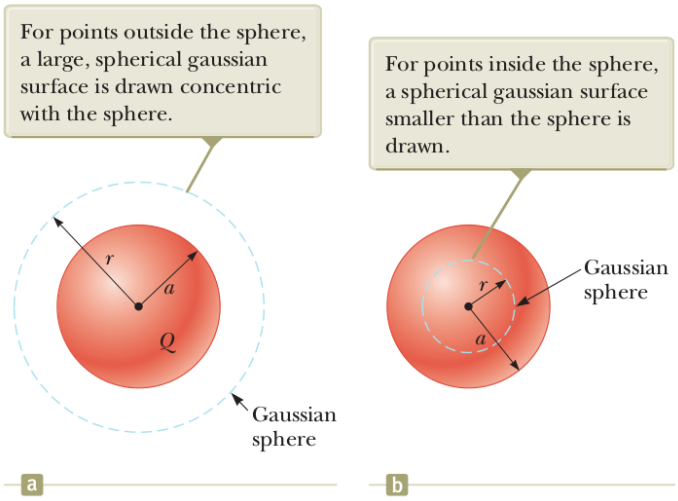


Figure 2.1: A uniformly charged insulating sphere of radius a and total charge Q. In diagrams such as this one, the dotted line represents the intersection of the Gaussian surface with the plane of the page.

Exercise 2.2 — 4 point. A sphere of radius R has a charge density $\rho = \rho_0(r/R)$ where ρ_0 is a constant and r is the distance from the center of the sphere.

(a) Calculate the total charge inside the sphere.

(b) Calculate the electric field everywhere.

■



3. Capacitance and dielectrics, metals

3.1 Capacitors

Any two conductors separated by an insulator (or a vacuum) form a capacitor.

Capacitance

The capacitance C of a capacitor is defined as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between the conductors.

$$C = \frac{Q}{\Delta V} \tag{3.1}$$

It is a measure of the ability of a capacitor to store energy. By definition capacitance is always a positive quantity. Named in honor of Michael Faraday, the SI unit of capacitance is the farad (F):

$$[C] = F = \frac{C}{V}$$

The farad is a very large unit of capacitance. In practice, typical devices have capacitances ranging from microfarads to picofarads.

3.2 Parallel-Plate Capacitors

Two parallel, metallic plates of equal area A are separated by a distance d. One plate carries a charge $+Q$, and the other carries a charge $-Q$. The surface charge density on each plate is $\sigma = \frac{Q}{A}$. What is it's capacitance?

The value of the electric field between the plates (as derived in Equation 2.13) is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \tag{3.2}$$

Since the field between the plates is uniform, the magnitude of the potential difference between the plates is

$$\Delta V = Ed = \frac{Qd}{\varepsilon_0 A} \tag{3.3}$$

From this the capacitance will be

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Qd}{\varepsilon_0 A}} = \varepsilon_0 \frac{A}{d} \tag{3.4}$$

The capacitance of a parallel-plate capacitor is

- proportional to the area of its plates and
- inversely proportional to the plate separation.

3.3 Combinations of Capacitors

3.3.1 Parallel combination of capacitors

Two capacitors connected as shown in Figure are known as a parallel combination of capacitors.

Let's call the maximum charges on the two capacitors Q_1 and Q_2.

$$Q_1 = C_1 \cdot \Delta V_1 Q_2 = C_2 \cdot \Delta V_2 \tag{3.5}$$

The total charge is Q_{tot}

$$Q_{tot} = Q_1 + Q_2 = C_1 \cdot \Delta V_1 + C_2 \cdot \Delta V_2 \tag{3.6}$$

Suppose you wish to replace these two capacitors by one equivalent capacitor C_{eq}.

$$Q_{tot} = C_{eq} \cdot \Delta V \tag{3.7}$$

then

$$Q_{tot} = C_{eq} \cdot \Delta V = C_1 \cdot \Delta V_1 + C_2 \cdot \Delta V_2 \tag{3.8}$$

from which it follows (as V_1 is the same as V_2 if we have a parallel combination) that

$$C_{eq} = C_1 + C_2 \tag{3.9}$$

Parallel combination of capacitors

In case of parallel combination of capacitors the equivalent capacitor will be the sum of the individual capacitors:

$$C_{eq} = \sum_i C_i \tag{3.10}$$

3.3.2 Series combination of capacitors

Two capacitors connected as shown in Figure are known as a series combination of capacitors.

The left plate of capacitor 1 and the right plate of capacitor 2 are connected to the terminals of a battery. The other two plates are connected to each other and to nothing else. Hence, they form an isolated system that is initially uncharged and must continue to have zero net charge!

When the battery is connected, electrons are transferred out of the left plate of C_1 to the right plate of C_2.

The negative charge leaving the left plate of C_2 causes negative charges to accumulate on the right plate of C_1.

As a result, both right plates end up with a charge $-Q$ and both left plates end up with a charge $+Q$.

Therefore, the charges on capacitors connected in series are the same

$$Q_1 = Q_2 = Q \tag{3.11}$$

where Q is the charge that moved between a wire and the connected outside plate of one of the capacitors.

$$\Delta V_{tot} = \Delta V_1 + \Delta V_2 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} = \frac{Q}{C_{eq}} \tag{3.12}$$

Therefore, the equivalent capacitor will have the capacitance based on

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{3.13}$$

Series combination of capacitors

In case of series combination of capacitors the equivalent capacitor will be the inverse of the sum of the inverse of the capacitance of the individual capacitors:

$$C_{eq} = \frac{1}{\sum_i \frac{1}{C_i}} \tag{3.14}$$

3.4 Energy stored in a capacitor

Because positive and negative charges are separated in the system of two conductors in a capacitor, electric potential energy is stored in the system.

Suppose q is the charge on the capacitor at some instant during the charging process.

At the same instant, the potential difference across the capacitor is

$$\Delta V = \frac{q}{C} \tag{3.15}$$

The work necessary to transfer an increment of charge dq from the plate carrying charge $-q$ to the plate carrying charge q

$$dW = \Delta V dq = \frac{q}{C} dq \tag{3.16}$$

then the work needed for the full process is

$$W = \int dW = \int_0^Q \frac{q}{C} dq \tag{3.17}$$

as C does not depend on dq we can move it out from the integral:

$$W = \frac{1}{C}\int_0^Q q dq = \frac{1}{C}\frac{Q^2}{2} \tag{3.18}$$

The potential energy stored in the capacitor will be the same as this work.

Energy stored in a capacitor

The energy stored in a capacitor

$$U_E = W = \frac{1}{C}\frac{Q^2}{2} \tag{3.19}$$

Using the definition of capacitance this is the same as

$$U_E = W = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2 \tag{3.20}$$

Let us put what we have learned for the parallel-plate capacitor

$$U_E = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)(Ed)^2 = \frac{1}{2}\varepsilon_0 A d E^2 \tag{3.21}$$

Let us define the energy density as the energy per unit volume

$$u_E = \frac{U_e}{V} = \frac{U_e}{Ad} \tag{3.22}$$

With this we get a formula that is generally valid regardless of the source of the electric field

Energy density

The energy stored in a capacitor

$$u_E = \frac{1}{2}\varepsilon_0 E^2 \tag{3.23}$$

3.5 Effect of dielectrics

A dielectric is a nonconducting material such as rubber, glass, or waxed paper. Consider a parallel-plate capacitor that without a dielectric has a charge Q_0 and a capacitance C_0 If a dielectric is now inserted between the plates the voltage between the plates decreases The capacitance increases by the factor κ

κ is called the dielectric constant It is dimensionless number that modifies the capacitance:

$$C = \kappa C_0 \tag{3.24}$$

3.6 Faraday cage

A Faraday cage is a metallic enclosure that prevents the entry or escape of an electromagnetic field (EM field). An ideal Faraday cage consists of an unbroken, perfectly conducting shell. This ideal cannot be achieved in practice, but can be approached by using fine-mesh copper screening. For best performance, the cage should be directly connected to an earth ground. Check out this (link) to see how an MIT professor plays with a Faraday cage.

3.7 Curvature effects

Let us have two metal ball connected with a wire. The first one has radius R_a and the second R_b. The fact that they are connected means that they are equipotential. By charging them up we have Q_A charges in the A ball and Q_B charges in the B ball. The potential will be on A

$$V_A = \frac{Q_A}{4\pi\varepsilon_0 R_A} = V_B = \frac{Q_B}{4\pi\varepsilon_0 R_B} \tag{3.25}$$

The consequence of this is that

$$\frac{Q_A}{R_A} = \frac{Q_B}{R_B} \tag{3.26}$$

This means that when the curvature of an object is higher the charges will accumulate there.

3.8 Questions

- Research the science of lightnings.
- What is St-Elmo's Fire?

3.9 Exercises

Exercise 3.1 — 2 point. A parallel-plate capacitor has a plate separation d and plate area A. An uncharged metallic slab of thickness a is inserted midway between the plates as shown on Figure 3.1. Calculate the capacitance of the system. ■

Exercise 3.2 — 2 point. Where must a current source be connected to the circuit diagram in Figure 3.2 in order to charge all the six capacitors having equal capacitances? ■

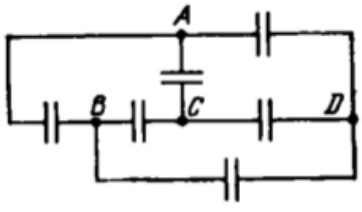


Figure 3.2: A combination of capacitors.

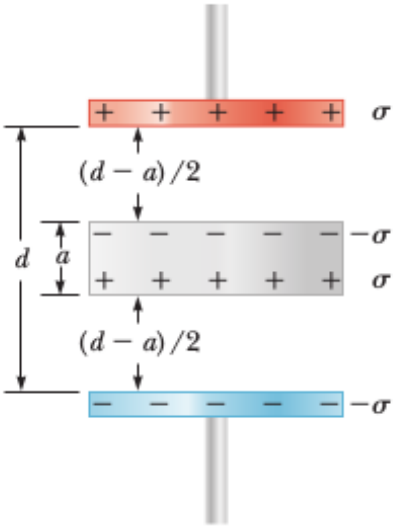


Figure 3.1: A parallel-plate capacitor of plate separation d partially filled with a metallic slab of thickness a.

Exercise 3.3 — 5 point. An insulated metal sphere of radius a with total charge q is placed inside a hollow grounded metal sphere of radius b. The center of the inner sphere is slightly displaced from the center of the outer sphere so that the distance between the two centers is δ

(a) Use the boundary conditions to determine the potential between the spheres in the case $\delta = 0$

(b) CHALLENGING: Find the charge distribution of the inner sphere and the force acting on it.

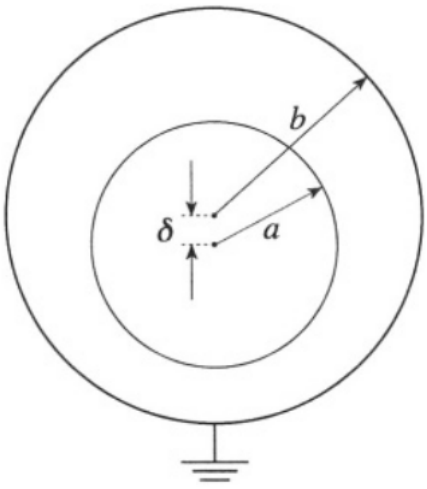


Figure 3.3: Not-so-concentric Spherical Capacitor.



4. Current, resistance and emf

4.1 Electric Current

Definition 4.1.1 — Current. Electric current is the rate of flow of charge If ΔQ is the amount of charge that passes through a surface in a time interval Δt, the average current I_{avg} is equal to the charge that passes through A per unit time:

$$I_{avg} = \frac{\Delta Q}{\Delta t} \tag{4.1}$$

If the rate at which charge flows varies in time, the current varies in time; we define the instantaneous current I as the limit of the average current as $\Delta t \to 0$:

$$I = \frac{dQ}{dt} \tag{4.2}$$

The SI unit of current is the ampere (A):

$$[I] = 1A = C/s$$

Consider a segment of a uniform conductor

$$dQ = qndV = qnAdx \tag{4.3}$$

Using the definition of velocity, we introduce the drift speed v_d:

$$dQ = qnAdx = qnAv_d dt \tag{4.4}$$

From this the current will become

$$I = qnAv_d \tag{4.5}$$

We can move on by defining the current density.

Definition 4.1.2 — Current density. The current density j in the conductor is defined as the current per unit area.

$$j = \frac{I}{A} = qnv_d \tag{4.6}$$

The unit of the current density

$$[j] = 1A/m^2$$

4.2 Resistance

A current density and an electric field are established in a conductor whenever a potential difference is maintained across the conductor. In some materials, the current density is proportional to the electric field. This is expressed by the (differential) Ohm's law.0

Ohm's law

For many materials (including most metals), the ratio of the current density to the electric field is a constant σ that is independent of the electric field producing the current.

$$j = \sigma E \tag{4.7}$$

where σ is called the conductivity of the conductor

Materials and devices that obey Ohm's are said to be ohmic. Experimentally, however, it is found that not all materials and devices have this property. Those are said to be nonohmic.

Ohm's law is not a fundamental law of nature; rather, it is an empirical relationship valid only for certain situations.

We can obtain an equation useful in practical applications as well. If the field is assumed to be uniform in the segment of straight wire, the magnitude of the potential difference across the wire is related to the field within the wire:

$$\Delta V = E \cdot L \tag{4.8}$$

Since $E = \frac{j}{\sigma}$ we get

$$\Delta V = \frac{j}{\sigma} \cdot L = \frac{L}{\sigma} \cdot j \tag{4.9}$$

Using that $j = \frac{I}{A}$

$$\Delta V = \frac{L}{\sigma} \cdot \frac{I}{A} = \frac{L}{\sigma A} I = R \cdot I \tag{4.10}$$

where we introduced the quantity $R = \frac{L}{\sigma A}$ which is called the resistance of the conductor.

Definition 4.2.1 We define the resistance as the ratio of the potential difference across a conductor to the current in the conductor:

$$R = \frac{\Delta V}{I} \tag{4.11}$$

This result shows that resistance has SI units of volts per ampere. One volt per ampere is defined to be one ohm (Ω)

$$[R] = 1\Omega = 1V/A$$

Most electric circuits use circuit elements called resistors to control the current in the various parts of the circuit.

Definition 4.2.2 The inverse of conductivity is resistivity ρ

$$\rho = \frac{1}{\sigma} \tag{4.12}$$

With this we can give another form of resistance:

$$R = \rho \frac{L}{A} \tag{4.13}$$

Every ohmic material has a characteristic resistivity that depends on the properties of the material and on temperature.

Over a limited temperature range, the resistivity of a conductor varies approximately linearly with temperature according to the expression

$$\rho = \rho_0[1 + \alpha(T - T_0)] \tag{4.14}$$

where ρ_0 is the resistivity at some reference temperature T_0 (usually taken to be 20 ° C) and α is the temperature coefficient of resistivity.

Temperature dependence of the resistance

With this the temperature dependence of the resistance

$$R = R_0[1 + \alpha(T - T_0)] \tag{4.15}$$

Please note that α could be negative as well. This behavior is indicative of a class of materials called semiconductor.

There is a class of metals and compounds whose resistance decreases to zero when they are below a certain temperature T_c , known as the critical temperature. These materials are known as superconductors.

4.3 Drude model

Consider a conductor as a regular array of atoms plus a collection of free electrons (conduction electrons). We identify the system as the combination of the atoms and the conduction electrons. The conduction electrons become free when the atoms condense into a solid.

In the absence of an electric field, the conduction electrons move in random directions through the conductor (electron gas).

When an electric field is applied to the system, the free electrons drift slowly in a direction opposite that of the electric field with an average drift speed v_d that is much smaller (typically 10^{-4} m/s) than their average speed v_{avg} between collisions (typically 10^6 m/s).

The electron's motion after a collision is independent of its motion before the collision. The excess energy acquired by the electrons due the work done on them by the electric field is transferred to the atoms of the conductor when the electrons and atoms collide.

The electron is a particle under a net force, and its acceleration can be found from Newton's second law

$$\mathbf{a} = \frac{\sum \mathbf{F}}{m} = \frac{q\mathbf{E}}{m_e} \tag{4.16}$$

Because the electric field is uniform, the electron's acceleration is constant, so the electron can be modeled as a particle under constant acceleration:

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t = \mathbf{v}_i + \frac{q\mathbf{E}}{m_e}t \tag{4.17}$$

where $\mathbf{v}_i$ is is the electron's initial velocity the instant after a collision and $\mathbf{v}_f$ is velocity immediately before the next collision occurs.

Assuming the initial velocities are randomly distributed over all possible directions the vector sum of $\mathbf{v}_i$ will become zero.

$$\mathbf{v}_{f,avg} = \mathbf{v}_d = \frac{q\mathbf{E}}{m_e}\tau \tag{4.18}$$

where τ is the average time interval between successive collisions.

4.4 Exercises

Exercise 4.1 — 2 point. A teapot with a surface area of $700\,cm^2$ is to be plated W with silver. It is attached to the negative electrode of an electrolytic cell containing silver nitrate $(Ag^+NO_3{}^-)$. The cell is powered by a 12.0-V battery and has a resistance of $1.80\,\Omega$.

(a) If the density of silver is $10.5 \cdot 10^3\,kg/m^3$, over what time interval does a 0.133-mm layer of silver build up on the teapot?

■

Exercise 4.2 — 2 point. Coaxial cables are used extensively for cable television and other electronic applications. A coaxial cable consists of two concentric cylindrical conductors. The region between the conductors is completely filled with polyethylene plastic as shown in Figure 4.1. Current leakage through the plastic, in the radial direction, is unwanted. (The cable is designed to conduct current along its length, but that is not the current being considered here.) The radius of the inner conductor is $a = 0.50\,cm$, the radius of the outer conductor is $b = 1.75\,cm$, and the length is $L = 15.0\,cm$. The resistivity of the plastic is $1.0 \cdot 10^{13}\,\Omega m$.

(a) Calculate the resistance of the plastic between the two conductors.

■

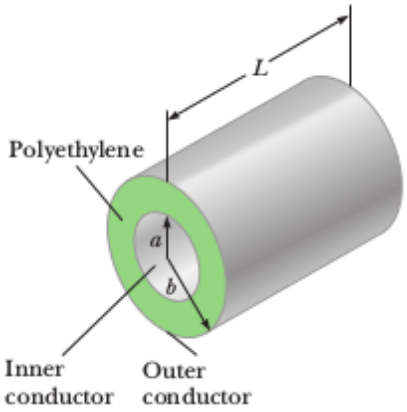
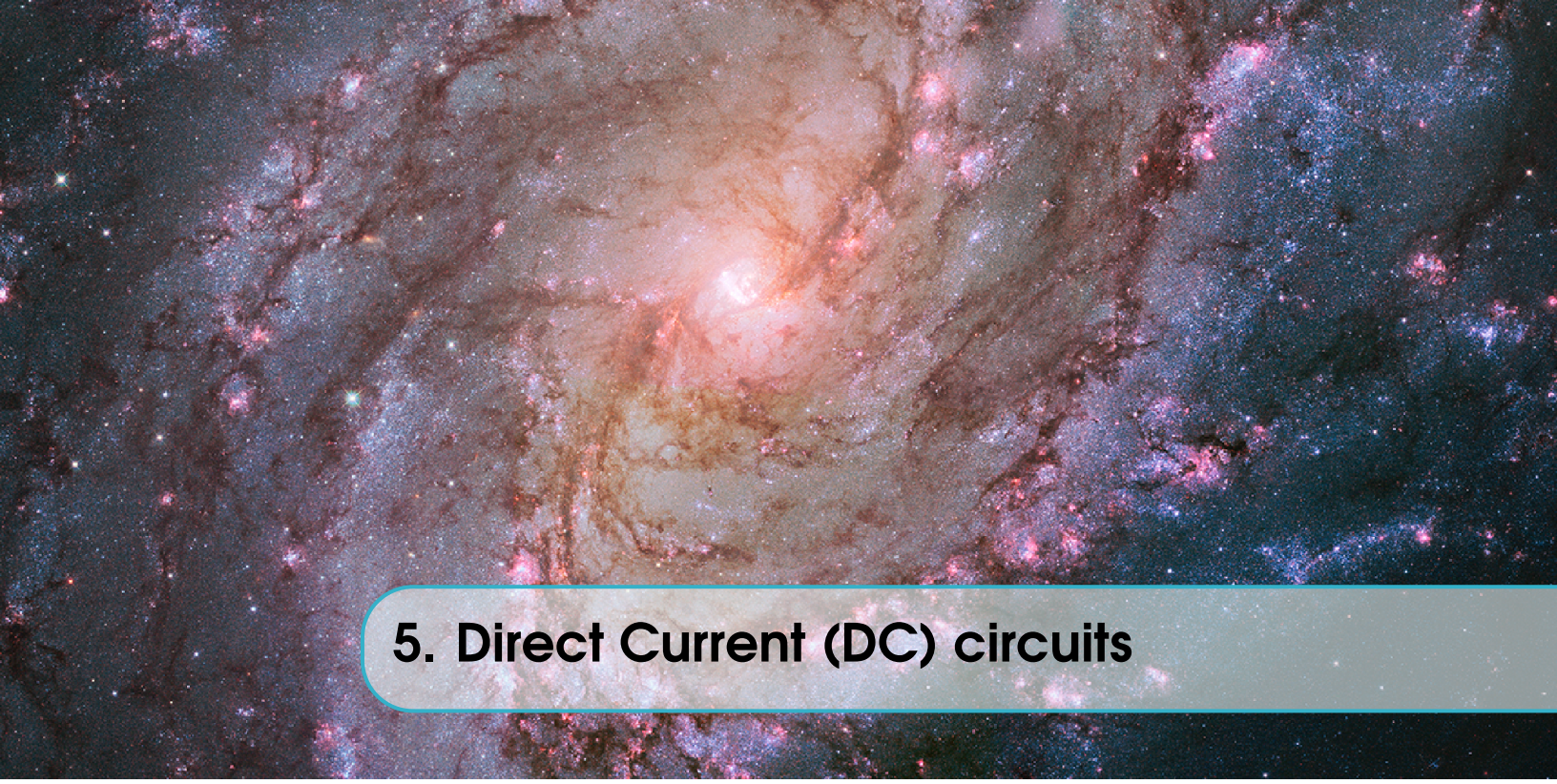


Figure 4.1: A coaxial cable.



5. Direct Current (DC) circuits

5.1 Electromotive force

Let us start with the definition of the direct current.

Definition 5.1.1 — Direct current. Direct current (DC) is the current in the circuit is constant in magnitude and direction.

A battery is called either a source of electromotive force or, more commonly, a source of emf.

Definition 5.1.2 The emf ε of a battery is the maximum possible voltage the battery can provide between its terminals.

Generally we can assume the connecting wires in a circuit have no resistance. The positive terminal of a battery is at a higher potential than the negative terminal. Because a real battery is made of matter, there is resistance to the flow of charge within the battery. This resistance is called internal resistance r.

An idealized battery has zero internal resistance. The potential difference across the battery (called its *terminal voltage*) equals its emf.

For a real battery, however, the terminal voltage is not equal to the emf for a battery in a circuit in which there is a current

$$\Delta V = \varepsilon - I \cdot r \tag{5.1}$$

An external resistance R, often called the load resistance can be included in the formula

$$\varepsilon = I \cdot R + I \cdot r \tag{5.2}$$

An the current like this is

$$I = \frac{\varepsilon}{R + r} \tag{5.3}$$

If R is much greater than r, as it is in many real-world circuits, we can neglect r.

5.2 Resistors in Series

When two or more resistors are connected together as on the Figure they are said to be in a series combination.

What if you wanted to replace the series combination with a single resistor?

In a series connection, if an amount of charge Q exits resistor R_1 , charge Q must also enter the second resistor R_2. Otherwise, charge would accumulate on the wire between the resistors. Therefore, the same amount of charge passes through both resistors in a given time interval and the currents are the same in both resistors:

$$I = I_1 = I_2 \tag{5.4}$$

The potential difference applied across the series combination of resistors divides between the resistors

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{5.5}$$

The potential difference across the battery is also applied to the equivalent resistance.

$$\Delta V = I \cdot R_{eq} \tag{5.6}$$

Then

$$IR_{eq} = I_1 R_1 + I_2 R_2 \tag{5.7}$$

resulting

$$R_{eq} = R_1 + R_2 \tag{5.8}$$

Resistance of resistors connected in series

The equivalent resistance of resistors connected in series

$$R_{eq} = \sum_i R_i \tag{5.9}$$

5.3 Resistors in parallel

Now consider two resistors in a parallel combination as shown in Figure. What if you wanted to replace the parallel combination with a single resistor?

Both resistors are connected directly across the terminals of the battery.

Therefore, the potential differences across the resistors are the same

$$\Delta V = \Delta V_1 = \Delta V_2 \tag{5.10}$$

When charges reach a junction (any such point in a circuit where a current can split), they split into two parts, with some going toward R_1 and the rest going toward R_2.

This split results in less current in each individual resistor than the current leaving the battery.

Because electric charge is conserved, the current I that enters point a must equal the total current leaving that point

$$I = I_1 + I_2 \tag{5.11}$$

Inserting the equivalent resistance

$$\frac{\Delta V}{R_{eq}} = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} \tag{5.12}$$

which then leads to the expression

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{5.13}$$

Resistance of resistors connected in parallel

The equivalent resistance of resistors connected in parallel will be given

$$R_{eq} = \frac{1}{\sum_i \frac{1}{R_i}} \tag{5.14}$$

5.4 Kirchhoff's Rules

Based on the fact that charge is conserved we can state the first Kirchhoff's Rule:

Junction rule

At any junction, the sum of the currents must equal zero

$$\sum_{junction} I = 0 \tag{5.15}$$

Due to the conservation of energy on the other hand, we have

Loop rule

The sum of the potential differences across all elements around any closed circuit loop must be zero

$$\sum_{closed\,loop} \Delta V = 0 \tag{5.16}$$

The following sign conventions apply when using the second rule

- Charges move from the high-potential end of a resistor toward the low-potential end, so if a resistor is traversed in the direction of the current: $\Delta V = -IR$
- If a resistor is traversed in the direction opposite the current: $\Delta V = +IR$
- If a source of emf is traversed in the direction of the emf (from negative to positive): $\Delta V = +\varepsilon$
- If a source of emf is traversed in the direction opposite the emf (from positive to negative): $\Delta V = -\varepsilon$

5.5 RC circuits

A circuit containing a series combination of a resistor and a capacitor is called an RC circuit.

How does the charge in the plate of the capacitor change over the time? Let us start with the situation that the plat is charged. Then it has its maximum charge. Later this charge will be less, however always proportional to the charge it started from. Change is Physics is denoted by a derivative so we are looking for a differential equation that has the form of

$$\frac{dQ(t)}{dt} = const \cdot Q(t) \tag{5.17}$$

What is the function that has this form? It is the exponential one, the constants should depend on R and C for sure, and we want a decaying function:

$$Q(t) = Q_{max}(1 - e^{-\frac{t}{RC}}) \tag{5.18}$$

How does the current change in the circuit?

$$I(t) = \frac{\varepsilon}{R} e^{\frac{t}{RC}}) \tag{5.19}$$

Definition 5.5.1 As RC has the unit of time we can introduce a characteristic time constant $\tau = RC$.

5.6 Exercises

Exercise 5.1 — 3 point. Consider the circuit shown in Figure 5.1. Each resistor has the resistance of R.

(a) Find the equivalent resistance between points a and b.

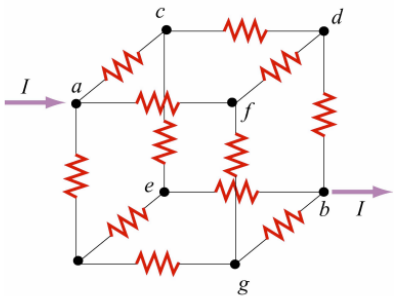


Figure 5.1

Exercise 5.2 — 4 point. Using Kirchhoff's rules:

(a) Find the current in each resistor shown in Figure 5.2.

(b) Find the potential difference between points c and f.

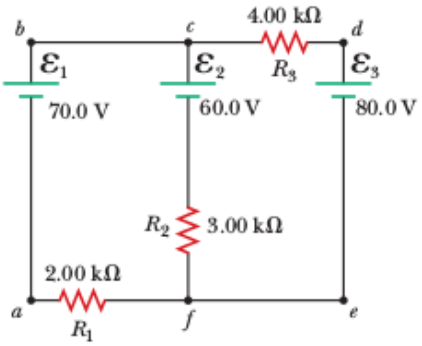


Figure 5.2

Exercise 5.3 — 4 point. Catherine spent her holiday in Egypt this year. After she returned, she soldered a pyramid composed of eight identical wires with linear resistivity ξ and length L, as shown in the picture below. What is the ratio of resistances between the pairs of vertices A–B and A–C?

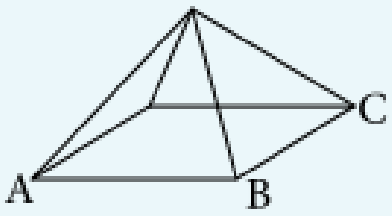


Figure 5.3

■

Exercise 5.4 — 4 point. A resistor pyramid is built in a branched structure as shown below. If there are N levels, constructed from individual resistances of value R, what is the total resistance R_N across the pyramid? What is the resistance of an infinite pyramid of this type? ■

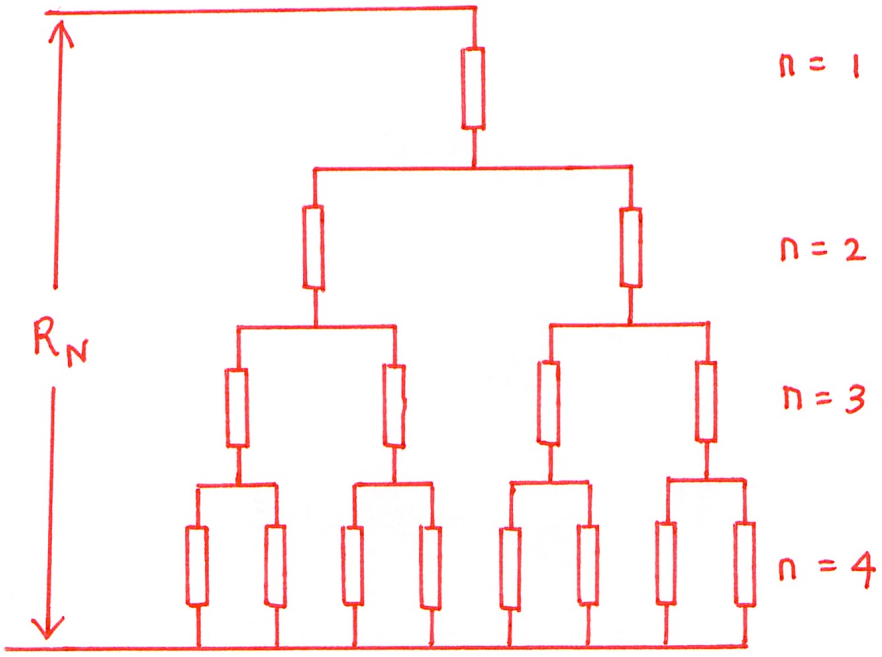


Figure 5.4

Exercise 5.5 — 4 point. Consider an infinite network of resistors of resistances R_0 and R_1 shown in Figure 5.5. Show that the equivalent resistance of this network is ■

$$R_{eq} = R_1 + \sqrt{R_1^2 + 2R_1R_0}\,.$$

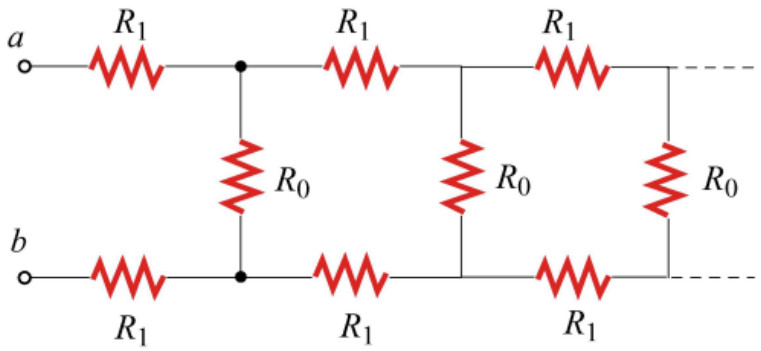
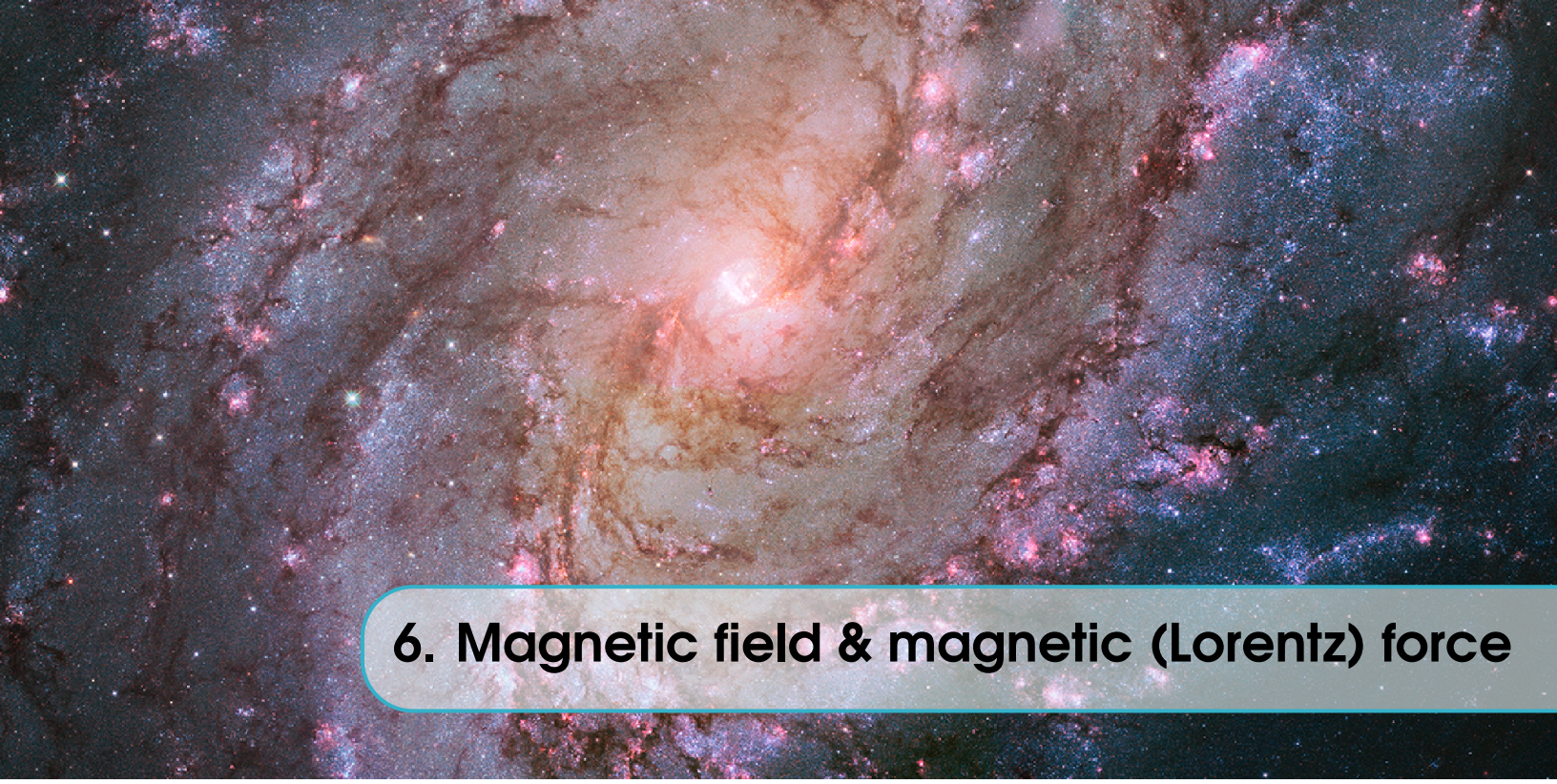


Figure 5.5



6. Magnetic field & magnetic (Lorentz) force

6.1 Brief history

The history of magnetic fields goes back to the Chinese, Arabic, Indian history in the 13th century already. The main theoretical developments started in the 19th century with the insights of Oersted, Faraday and Henry. A crucial step was done by Maxwell in the 20th century. He united the electric and the magnetic forces and predicted that the light is a wave in the EM field. The quantum theory of EM was developed by Feynman, Schwinger and Dyson still in the 20th century. Next step was to unite the EM field with the weak field, which was done by Weinberg, Salam and Glashow still in the 20th century.

6.1.1 Magnetic field

An example of a magnetic field is that surrounds a magnetic substance making up a permanent magnet. Historically, the symbol **B** has been used to represent a magnetic field. We can visually represent the magnetic field with magnetic field lines.

6.2 Magnetic force

Experiments show that the magnetic force is proportional to

- the charge q of the particle.
  - The magnetic force on a negative charge is directed opposite to the force on a positive charge moving in the same direction.
- the magnitude of the magnetic field vector **B**
- the speed v of the particle.
  - If the velocity vector makes an angle u with the magnetic field, the magnitude of the magnetic force is proportional to sin u
  - When a charged particle moves parallel to the magnetic field vector, the magnetic force on the charge is zero.

These observations can be summarized in a compact way by writing the magnetic force in the form.

Magnetic force

The force acting on a charge q having the the speed v in a **B** magnetic field can be calculated as

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \tag{6.1}$$

Please note that the right hand rule is applied.

The unit is going to be

$$[B] = \frac{N}{C \cdot m/s} = \frac{N}{Am} = 1T$$

where the T stands for tesla. Please note that 1 T is a very big number. The magnitude of Earth's magnetic field at the surface ranges from 25 to 65 microteslas. The magnets in the MRI/NMR machines is about 5 Tesla. The magnetic field in CERN's LHC is 8 T.

6.3 Particle in a Uniform Magnetic Field

A particle in a uniform magnetic field will (by the constant Lorentz force) move on a cyclical path. The particle will experience centripetal acceleration. From earlier studies we know that the centripetal force can be calculated as

$$|\mathbf{F}| = \frac{mv^2}{r} \tag{6.2}$$

Equating this with the Lorentz force we can calculate the radius of the circular path:

$$r = \frac{mv}{qB} \tag{6.3}$$

In case there is a perpendicular velocity involved then the particle will move on a helical path. The angular speed of the particle cyclotron frequency is going to be

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{6.4}$$

and the angular time is

$$T = 2\pi\frac{r}{v} = \frac{2\pi m}{qB} \tag{6.5}$$

6.4 Applications

Mass spectrometer works the following way:

- Positively charged particles are sent first through a velocity selector (a capacitor for example)
- then into a region where the magnetic field B causes the particles to move in a semicircular path and strike a detector array at P.
- thus fragments of a material are separated by their masses (essentially the mass/charge ratios)

6.5 Magnetic flux

Definition 6.5.1 We define the magnetic flux similarly to the electric fields

$$\Phi_B = \int B_\perp dA = \int B cos\phi dA = \int \mathbf{B}\cdot\mathbf{dA} \tag{6.6}$$

The unit for this is

$$[\Phi_B] = Tm^2 = 1W$$

where W is the weber.

Calculating this on a closed surface we get the Gauss's law for magnetism:

Gauss's law for magnetism

$$\oint_{\mathscr{A}} \mathbf{B}\cdot\mathbf{dA} = 0 \tag{6.7}$$

The total flux through a closed surface is always zero.

The meaning of this is that there is no magnetic monopole!

Please note that in any modern quantum theory this is violated. Without a magnetic monopole the quantization of the electric charge cannot be explained.

6.6 Questions

- What is a magnetic monopole? What theories include it? How can we search for it?

6.7 Exercises

Exercise 6.1 — 2 point. An electron gun in a cathode ray tube accelerates an electron with mass m and charge $-e$ across a potential difference of 50 V and directs it horizontally towards a fluorescent screen 0.4 m away. How far does the electron fall during its journey to the screen? Take $m \approx 10^{-30}kg$ and $e = 1.6\cdot 10^{-19}C$

■

Exercise 6.2 — 2 point. In Niels Bohr's 1913 model of the hydrogen atom, an electron circles the proton at a distance of $5.29\cdot 10^{-11}$ m with a speed of $2.19\cdot 10^6$ m/s.

(a) Compute the magnitude of the magnetic field this motion produces at the location of the proton.

■

Exercise 6.3 — 2 point. Two infinitely long, parallel wires are lying on the ground a distance a = 1.00 cm apart. A third wire, of length L = 10.0 m and mass $400\,g$, carries a current of $I_1 = 100\,A$ and is levitated above the first two wires, at a horizontal position midway between them. The infinitely long wires carry equal currents I_2 in the same direction, but in the direction opposite that in the levitated wire.

(a) What current must the infinitely long wires carry so that the three wires form an equilateral triangle?

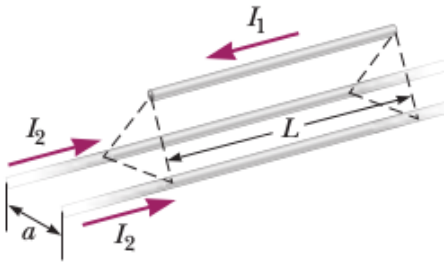


Figure 6.1: A triangle of wires

Exercise 6.4 — 2 point. Magnetic Field Determine the initial direction of the deflection of charged particles as they enter the magnetic fields shown in Figure 6.2

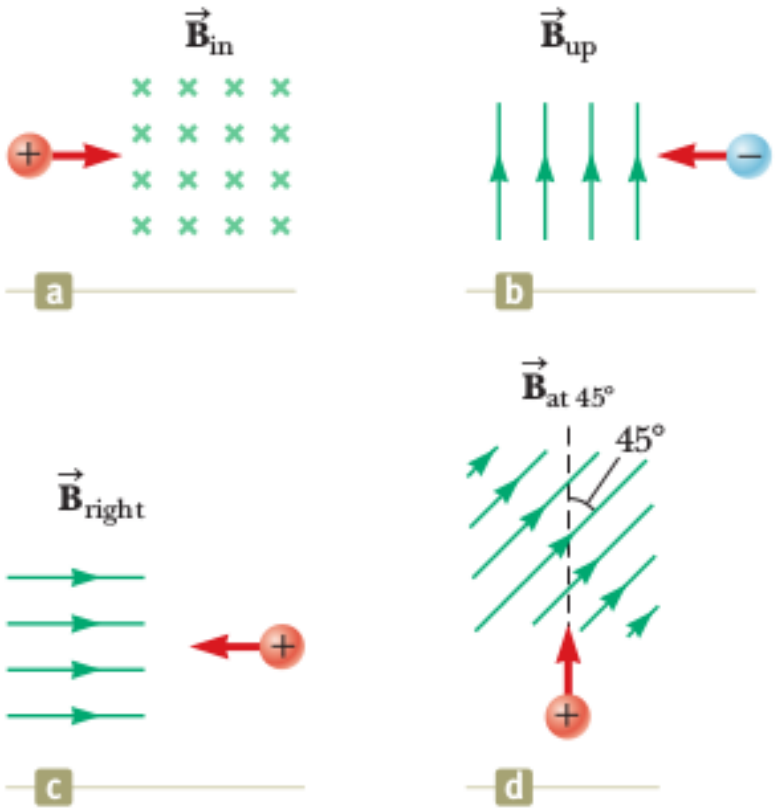


Figure 6.2

Exercise 6.5 — 4 point. A proton (charge $+e$, mass m_p), a deuteron (charge $+e$, mass $2m_p$), and an alpha particle (charge $+2e$, mass $4m_p$) are accelerated from rest through a common potential difference ΔV. Each of the particles enters a uniform magnetic field $\mathbf{B}$, with its velocity in a direction perpendicular to $\mathbf{B}$. The proton moves in a circular path of radius r_p.

In terms of r_p, determine

(a) the radius r_d of the circular orbit for the deuteron and

(b) the radius r_α for the alpha particle.

■

Exercise 6.6 — 2 point. Figure 6.3. shows a view of a flat surface with area of $3.00 cm^2$ in a uniform magnetic field. If the magnetic flux through this area is $0.90 mWb$,

(a) calculate the magnitude of the magnetic field and

(b) find the direction of the area vector (**dA**).

■

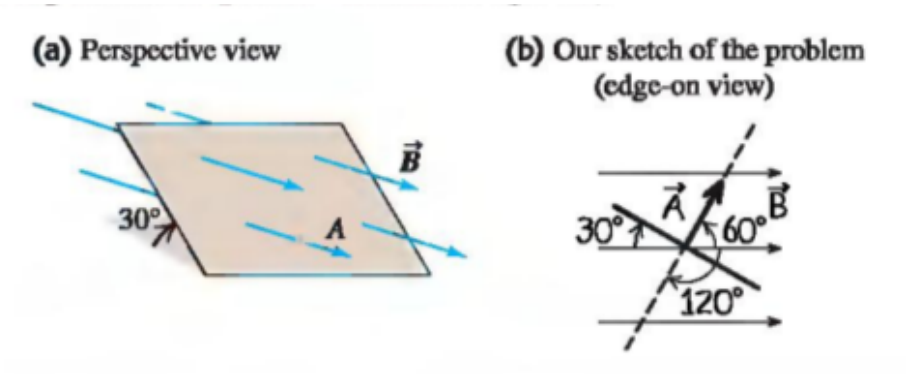


Figure 6.3



7. Induction, Faraday's Law & Lenz's Law

7.1 Biot–Savart Law

From experimental results, Biot and Savart arrived at a mathematical expression a magnetic field dB at a point P associated with a length element ds of a wire carrying a steady current I

- is perpendicular both to ds and to the unit vector r directed from ds toward P
- is inversely proportional to r^2 , where r is the distance from ds to P
- is proportional to the current I

These observations are summarized in the mathematical expression known today as the Biot–Savart law

Infinitesimal Biot–Savart law

$$\mathbf{dB} = \frac{\mu_0}{4\pi} I \frac{\mathbf{ds} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} \tag{7.1}$$

where $\hat{\mathbf{r}}$ is the unit vector, the μ_0 is a constant called the **permeability of free space**:

$$\mu_0 = 4\pi \cdot 10^{-7} Tm/A$$

Integrating the infinitesimal

Biot–Savart law

$$\mathbf{B} = \int \mathbf{dB} = \frac{\mu_0 I}{4\pi} \int \frac{\mathbf{ds} \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} \tag{7.2}$$

7.2 Magnetic field around a wire

Consider a thin, straight wire of finite length carrying a constant current I and placed along the x axis as shown in the Figure. Determine the magnitude and direction of the magnetic field at point P due to this current.

First we evaluate the cross product in the Biot–Savart law

$$\mathbf{ds} \times \hat{\mathbf{r}} = |\mathbf{ds} \times \hat{\mathbf{r}}|\mathbf{k} = |\mathbf{ds}||\hat{\mathbf{r}}| sin\left(\frac{\pi}{2} - \theta\right)\mathbf{k} \tag{7.3}$$

Using the trigonometric knowledge about sin() and cos()

$$\mathbf{ds} \times \mathbf{r} = dxcos(\theta)\mathbf{k} \tag{7.4}$$

where we used that $|\mathbf{ds}| = dx$ and that $|\hat{\mathbf{r}}| = 1$.

Substitute into the Biot-Savart Eq

$$\mathbf{dB} = db\mathbf{k} = \frac{\mu_0 I}{4\pi}\frac{1}{|\mathbf{r}|^2}dxcos(\theta)\mathbf{k} \tag{7.5}$$

From the geometry we can learn that

$$r = \frac{a}{cos\theta} \tag{7.6}$$

and

$$x = -atan\theta \tag{7.7}$$

Differentiating the later we get

$$x = -\frac{a}{cos^2\theta}d\theta \tag{7.8}$$

Putting this back:

$$dB = -\frac{\mu_0 I}{4\pi}\frac{1}{\left(\frac{a}{cos\theta}\right)^2}\frac{a}{cos^2\theta}d\theta cos(\theta) = -\frac{\mu_0 I}{4\pi a}cos\theta d\theta \tag{7.9}$$

from which

$$B = \int dB = -\frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2} cos\theta d\theta \tag{7.10}$$

Consider the special case of an infinitely long, straight wire! Then $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{-\pi}{2}$

With this we have

$$\int_{\theta_1}^{\theta_2} cos\theta d\theta = sin\left(\frac{\pi}{2}\right) - sin\left(-\frac{\pi}{2}\right) = 2 \tag{7.11}$$

Inserting this back:

$$B = -\frac{\mu_0 I}{4\pi a}\cdot(-2) = \frac{\mu_0 I}{2\pi a} \tag{7.12}$$

Magnetic field around a wire

In a point r far from a wire carrying I current the magnetic field

$$B = \frac{\mu_0 I}{2\pi r} \tag{7.13}$$

7.3 The magnetic force between two parallel conductors

The formula to calculate the magnetic force between two parallel conductors is given below

Magnetic Force Between Two Parallel Conductors

Having a wire with ℓ length, carrying current I, in a magnetic field B will create a force

$$F = BI\ell \tag{7.14}$$

or in vectorial form

$$\mathbf{F} = -(\mathbf{B} \times \mathbf{j})\ell \tag{7.15}$$

where – a reminder – the current density relates to the current as $I = \int \mathbf{jdA}$.

We just calculated the magnetic field created by I current in a wire:

$$B = \frac{\mu_0 I}{2\pi r}$$

Inserting this, we have

$$F_1 = B_2 I_1 \ell = \frac{\mu_0 I_2}{2\pi r} I_1 \ell \tag{7.16}$$

Calculating it by unit length:

$$\frac{F_1}{\ell} = \frac{\mu_0 I_2 I_1}{2\pi r} \tag{7.17}$$

The force between two parallel wires is used to define the ampere.

Definition 7.3.1 When the magnitude of the force per unit length between two long, parallel wires that carry identical currents and are separated by 1 m is $2 \cdot 10^{-7} N/m$, the current in each wire is defined to be 1 A.

7.4 Ampère's Law

Magnetic field lines have no beginning and no end, they rather form a closed loop. That is a major difference between magnetic field lines and electric field lines.

Let's calculate the path integral over a current in a circular path.

$$\oint \mathbf{B} \cdot \mathbf{ds} = B \oint ds = \frac{\mu_0 I}{2\pi r} \cdot (2\pi r) = \mu_0 I \tag{7.18}$$

This leads to the Ampère's Law:

Ampère's Law

The line integral of $\mathbf{B} \cdot \mathbf{ds}$ around any closed path equals $\mu_0 I$, where I is the total steady current passing through any surface bounded by the closed path

$$\oint \mathbf{B} \cdot \mathbf{ds} = \mu_0 I \tag{7.19}$$

7.5 Magnetism in Matter

We have three categories for Magnetism in Matter:

- Ferromagnetism
  Once the moments are aligned, the substance remains magnetized after the external field is removed
- Paramagnetism
  Atomic moments tend to line up with the field. After the field is removed the spins will be randomized
- Diamagnetism
  A weak magnetic moment is induced in the direction opposite the field

7.6 Faraday's (and Lenz's) Law of Induction

So far, our studies in electricity and magnetism have focused on the electric fields produced by stationary charges and the magnetic fields produced by moving charges. Now we will explore the effects produced by magnetic fields that vary in time.

Faraday's Law of Induction

If you move a magnet in a wire we experience that emf is induced in a loop when the magnetic flux through the loop changes with time. It is directly proportional to the time rate of change of the magnetic flux through the loop:

$$\varepsilon = \frac{d\Phi_B}{dt} \tag{7.20}$$

If a coil consists of N loops with the same area and the magnetic flux through one loop emf is induced in every loop. The loops are in series, so their emfs add; therefore, the total induced emf in the coil is given by

$$\varepsilon = N\frac{d\Phi_B}{dt} \tag{7.21}$$

Lenz's Law

The induced current in a loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through the area enclosed by the loop.

$$\varepsilon = -\frac{d\Phi_B}{dt} \tag{7.22}$$

We have seen that a changing magnetic flux induces an emf and a current in a conducting loop. The induction of a current in the loop implies the presence of an induced electric field, which must be tangent to the loop.

The work done by the electric field in moving a charge q once around the loop is equal to q time emf

$$q\varepsilon = qE(2\pi r) \tag{7.23}$$

from which

$$E = \frac{\varepsilon}{2\pi r} \tag{7.24}$$

Given that $\Phi_B = BA = B\pi r^2$ the induced electric field can be calculated as

$$E = -\frac{1}{2\pi r}\frac{d\Phi_B}{dt} = -\frac{r}{2}\frac{dB}{dt} \tag{7.25}$$

The induced electric field is a nonconservative field that is generated by a changing magnetic field.

Faraday's law of induction can be written in the general form by using that $\varepsilon = \oint \mathbf{Eds}$

Faraday's/Lenz's Law of Induction

The line integral of the electric field around a closed loop is equal to the negative of the rate of change of the magnetic flux through the area enclosed by the loop.

$$\oint \mathbf{Eds} = \frac{d\Phi_B}{dt} \tag{7.26}$$

7.7 Exercises

Exercise 7.1 — 2 point. A coil consists of 200 turns of wire. Each turn is a square of side d = 18 cm, and a uniform magnetic field directed perpendicular to the plane of the coil is turned on.

(a) If the field changes linearly from 0 to 0.50 T in 0.80 s, what is the magnitude of the induced emf in the coil while the field is changing?

■



8. EM waves & intro to modern physics

8.1 Maxwell's Equations

The first law is

Gauss's law for electricity

$$\oint_{\mathscr{A}} \mathbf{E} \cdot \mathbf{dA} = \frac{Q_{inside}}{\varepsilon_0} \tag{8.1}$$

the second

Gauss's law for magnetism

$$\oint_{\mathscr{A}} \mathbf{B} \cdot \mathbf{dA} = 0 \tag{8.2}$$

and then we have the

Faraday's-Lenz's law for electricity

$$\oint_{\mathscr{A}} \mathbf{E} \cdot \mathbf{ds} = -\frac{d\Phi_B}{dt} \tag{8.3}$$

and

Ampère's Law

$$\oint_{\mathscr{A}} \mathbf{B} \cdot \mathbf{ds} = \mu_0 I \tag{8.4}$$

This latest one was extended by Maxwell to

Maxwell-Ampère's Law

$$\oint_{\mathscr{A}} \mathbf{B}\cdot\mathbf{ds} = \mu_0 I + \varepsilon\mu\frac{d\Phi_E}{dt} \tag{8.5}$$

in case we have a changing electric field.
The only thing we are missing is the force law:

Lorentz's force

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B} \tag{8.6}$$

Using the Maxwell equations and the Lorentz force law one can describe all electromagnetic phenomena.

8.2 Differential Maxwell's Equations

By considerations in the Appendix we can transform these to the differential Maxwell equations.

Differential Gauss's law for electricity

$$div(\mathbf{E}) = \frac{\rho_{inside}}{\varepsilon_0} \tag{8.7}$$

the second

Differential Gauss's law for magnetism

$$div(\mathbf{B}) = 0 \tag{8.8}$$

and then we have the

Differential Faraday's-Lenz's law for electricity

$$rot(\mathbf{E}) = -\frac{d\mathbf{B}}{dt} \tag{8.9}$$

Differential Maxwell-Ampère's Law

$$rot(\mathbf{B} = \mu_0\mathbf{j} + \varepsilon\mu\frac{d\mathbf{E}}{dt} \tag{8.10}$$

8.3 EM Waves

Let us consider the EM fields in vacuum. This means there are no charges or currents: $\rho = 0$ and $\mathbf{j} = 0$
So we have

$$div(\mathbf{E}) = 0 \tag{8.11}$$

and

$$div(\mathbf{B}) = 0 \tag{8.12}$$

and

$$rot(\mathbf{E}) = -\frac{d\mathbf{B}}{dt} = -\partial_t \mathbf{B} \tag{8.13}$$

and

$$rot(\mathbf{B}) = \varepsilon\mu\frac{d\mathbf{E}}{dt} = \varepsilon\mu\partial_t \mathbf{E} \tag{8.14}$$

Taking the curl of the last equation and using the mathematical identity that $rot(rot\mathbf{F}) = grad(div(\mathbf{F})) - \triangle\mathbf{F}$ we have

$$rot(rot(\mathbf{B})) = grad(div(\mathbf{B})) - \triangle\mathbf{B} = \varepsilon\mu rot(\partial_t \mathbf{E}) = \varepsilon\mu\partial_t(rot(\mathbf{E})) = -\varepsilon\mu\partial_t^2(\mathbf{B}) \tag{8.15}$$

On the right hand side we used that the derivative by the time and the derivative by the space can be exchanged and the Faraday-Lenz law. As a next step let us remember that in the Maxwell theory the Gauss law for magnetism will zero out the div(**B**) term:

$$-\triangle\mathbf{B} = -\varepsilon\mu\partial_t^2(\mathbf{B}) \tag{8.16}$$

Putting it to the left hand side

$$\triangle\mathbf{B} - \varepsilon\mu\partial_t^2(\mathbf{B}) = 0 \tag{8.17}$$

The same thing can be repeated for the electric case and we have:

$$\triangle\mathbf{E} - \varepsilon\mu\partial_t^2(\mathbf{E}) = 0 \tag{8.18}$$

The 1+1 dimensional case will be

$$\partial_x^2\mathbf{B} - \varepsilon\mu\partial_t^2(\mathbf{B}) = 0 \tag{8.19}$$

and

$$\partial_x^2\mathbf{E} - \varepsilon\mu\partial_t^2(\mathbf{E}) = 0 \tag{8.20}$$

These are second ordered differential equation for which the solution is the sinus and the cosine functions:

$$\mathbf{E} = \mathbf{E}_{max}cos(kx - \omega t) \tag{8.21}$$

and

$$\mathbf{B} = \mathbf{B}_{max}cos(kx - \omega t) \tag{8.22}$$

where the k is the wave number:

$$k = \frac{2\pi}{\lambda} \tag{8.23}$$

where the λ is the wave lenght.

The ω is the angular frequency:

$$\omega = 2\pi f \tag{8.24}$$

where f is the frequency.

We see that the vectors of **E** and **B** are perpendicular to each other and also to the direction of the movement as well.

Interestingly the value of the

$$\mu_0\varepsilon_0 = \frac{1}{c^2} \tag{8.25}$$

From this we learn that the oscillation of the electromagnetic field is the light.

8.4 Relationship between the magnitudes of B and E in an EM wave

Let us consider the waves as plane waves. Then applying the Faraday law

$$\oint_{\mathscr{A}} \mathbf{E}\cdot\mathbf{ds} = -\frac{d\Phi_B}{dt} \tag{8.26}$$

we have:

$$\oint_{\mathscr{A}} \mathbf{E}\cdot\mathbf{ds} = -\mathbf{Eda}+0+0+0 \tag{8.27}$$

and

$$-\frac{d\Phi_B}{dt} = -\frac{\mathbf{BdA}}{dt} = -\frac{Bacdt}{dt} = -Bac \tag{8.28}$$

so we have

$$-Ea+0+0+0 = -Bac \tag{8.29}$$

from which

Relationship between the magnitudes of B and E in an EM wave

$$E = Bc \tag{8.30}$$

8.5 Energy Carried by Electromagnetic Waves

The magnitude of the Poynting vector represents the rate at which energy passes through a unit surface area perpendicular to the direction of wave propagation.

Therefore, the magnitude of it represents power per unit area.

The direction of the vector is along the direction of wave propagation.

Definition 8.5.1 — Poynting vector. The Poynting vector is defined as

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{8.31}$$

Time average of S over one or more cycles is called the wave intensity

Definition 8.5.2

$$I = S_{avg} = \frac{E_{max}B_{max}}{2\mu_0} = \frac{E_{max}^2}{2\mu_0} = \frac{cB_{max}^2}{2\mu_0} \tag{8.32}$$

The performance of the beam will be

$$P = S_{avg}A = \int \mathbf{SdA} \tag{8.33}$$

8.6 Maximum speed in the universe

How fast can the energy of an EM wave travel?

Let us use units where the permittivity and the permeability is 1.

The energy density was calculated as

$$w = \frac{1}{2}\left(E^2+B^2\right)) \tag{8.34}$$

The energy change volume will be then

$$\Delta E_{nergy} = w\Delta V = wA\Delta L = wAu\Delta t \tag{8.35}$$

where we expressed the volume using this maximum speed u.
The absolute value of the Poynting vector gives the energy change in a unit time in a unit area:

$$|\mathbf{S}| = \frac{\Delta E_{nergy}}{A\Delta t} \tag{8.36}$$

so the energy change is like

$$\Delta E_{nergy} = |\mathbf{S}|A\Delta t \tag{8.37}$$

Putting it together

$$\Delta E_{nergy} = |\mathbf{S}|A\Delta t = wAu\Delta t \tag{8.38}$$

So we have

$$u = \frac{|\mathbf{S}|}{w} \tag{8.39}$$

Now inserting previous definitions

$$u = \frac{|\mathbf{S}|}{w} = \frac{c|\mathbf{E}\times\mathbf{B}|}{\frac{1}{2}(E^2+B^2)} = 2c\frac{|\mathbf{E}\times\mathbf{B}|}{(E^2+B^2)} = 2c\frac{|\mathbf{E}|\cdot|\mathbf{B}|\cdot cos(\phi)}{(E^2+B^2)} \tag{8.40}$$

Now let us use that the cosine function is limited

$$cos(\phi) \leq \pm 1 \tag{8.41}$$

and that

$$\sqrt{xy} \leq \frac{x+y}{2} \tag{8.42}$$

Applying

$$u \leq 2c\frac{|\mathbf{E}|\cdot|\mathbf{B}|}{(E^2+B^2)} = 2c\frac{\sqrt{|\mathbf{E}|^2\cdot|\mathbf{B}|^2}}{(E^2+B^2)} \leq 2c\frac{E^2+B^2}{2(E^2+B^2)} = c \tag{8.43}$$

So the maximum speed with which energy can travel is the speed of light. With this we enter topics for Modern Physics: speacial relativity, quantum mechanics and finally quantum field theories.

8.7 Exercises

Exercise 8.1 — 2 point. Given a cylindering capacitor in a magnetic field what will be the direction of the Pointing vector? The magnetic field points in the longitudinal direction. What would be the consequence of this? What will happen if we connect the two plates of the capacitor? ■

Exercise 8.2 — 3 point. In SI units, the electric field in an electromagnetic wave is described by

$$E_y = 100\,sin(1.00\cdot 10^7 x - \omega t)$$

Find
(a) the amplitude of the corresponding magnetic field oscillations,
(b) the wavelength,
(c) and the frequency.

■

Exercise 8.3 — 5 point. Two opposite walls of a rigid box are uniformly charged with surface charge densities σ and $-\sigma$ respectively. The positively charged wall occupies the region $0 \leq x \leq a$, $0 \leq y \leq b$ of the plane $z = h$, while the negatively charged wall occupies the region $0 \leq x \leq a$, $0 \leq y \leq b$ of the $x-y$ plane. Inside the box, there is a uniform magnetic field $\mathbf{B} = (0, B_0, 0)$. Assume that h is much smaller than both a and b and that the charged walls are nonconducting.

(a) Estimate the impulse experienced by the box when the magnetic field is switched off.

(b) Show that it is equal to the initial momentum of the electromagnetic field.

Hint: The initial momentum may be found from the Poynting vector. ■

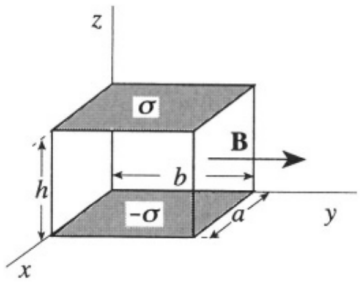


Figure 8.1: A box



A. Natural units

The three relevant quantities in a kinematic process are length (L), mass (M) and time (T). The SI units are chosen to be convenient for the everyday life, but they are not convenient at all for the relativistic quantum description of the world.

In the natural units the fundamental constants of this theory, c and $\hbar$, are both equal to one.

$$\hbar = c = 1$$

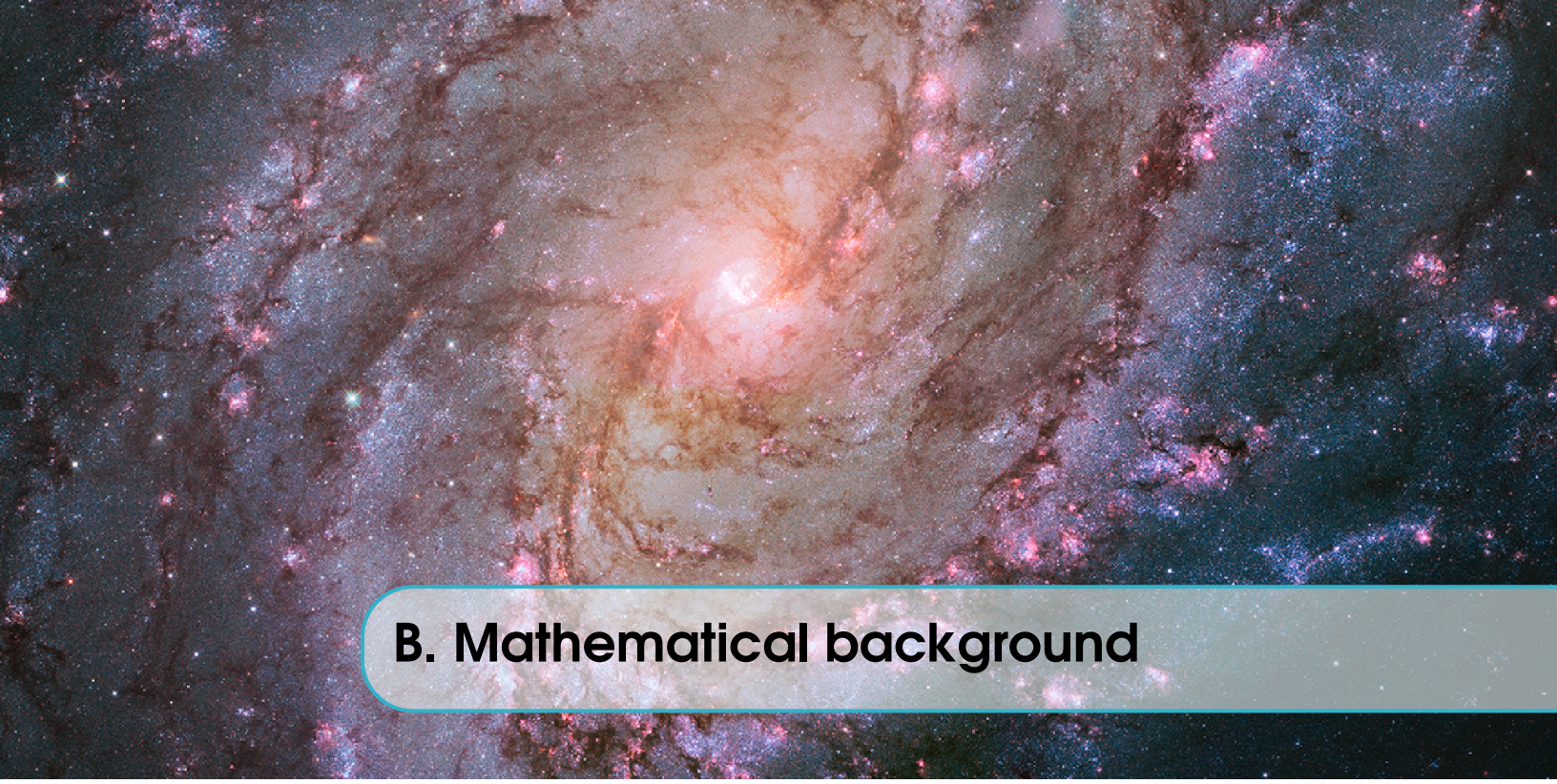
Since

$$[\hbar] = ML^2T^{-1} \text{ and } [c] = LT^{-1}$$

the have a system like this we need to fix the unit of energy (ML^2T^{-2}). For this we choose eV as a unit, that is the energy of an electron in a capacitor with 1 V of voltage on it. This is very practical in atomic physics but the particle physics scales are rather in the range of the GeV.

As a consequence of this everything will be measured as powers of the GeV. Some examples can be found here:

$$\begin{aligned} 1 \text{ kg} &= 5.61 \cdot 10^{26}\, GeV \\ 1 \text{ s} &= 1.52 \cdot 10^{24}\, GeV^{-1} \\ 1 \text{ m} &= 5.07 \cdot 10^{15}\, GeV^{-1} \end{aligned}$$



B. Mathematical background

B.1 Functions and derivatives

Given a function $f(x)$ and a small deviation from x denoted with ε the value of the function in the $x+\varepsilon$ position can be approximated using the Taylor expansion:

$$f(x+\varepsilon) = f(x) + \varepsilon \frac{\partial f}{\partial x} + \mathcal{O}(\varepsilon^2) \tag{B.1}$$

where $\mathcal{O}$ is the ordo symbol which means that we neglect the terms with the given order. The two dimensional version of this is given by

$$f(x+\varepsilon_x, y+\varepsilon_y) = f(x,y) + \varepsilon_x \frac{\partial f}{\partial x} + \varepsilon_y \frac{\partial f}{\partial y} + \mathcal{O}(\varepsilon^2) \tag{B.2}$$

B.2 Vectors

The absolute value of the dot product of two vectors **a** and **b** is calculated as

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot cos(\theta) \tag{B.3}$$

while the absolute value of the cross product is

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{c}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot sin(\theta) \tag{B.4}$$

where θ is the angle between two vectors.

B.3 Fields

A field is fundamental algebraic structure that we can imagine as a map from a space to a value. We differentiate three kind of them now:

- *Scalar field*: It associates a scalar value to every point in a space. Examples: temperature distribution throughout space, the pressure distribution in a fluid or the Higgs field.

- *Vector field*: It assigns a vector to each point in a subset of space. It can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane. Example: the velocity of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.
- *Tensor field*: It associates a tensor to every point in a space. Example: In Einstein's General Relativity the curvature of spacetime changes throughout space.

B.4 Vector algebra

Vector fields satisfy the usual linear algebraic operations:

- Vector addition: Addition of two vector fields yields to a vector field.

$$\mathbf{v}_1(\mathbf{r}) + \mathbf{v}_2(\mathbf{r}) = \mathbf{v}_3(\mathbf{r}) \tag{B.5}$$

- Scalar multiplication: Multiplication of a scalar field and a vector field yields to a vector field.

$$c(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}) = \mathbf{w}(\mathbf{r}) \tag{B.6}$$

- Dot product: The dot product of two vector field yields to a scalar field.

$$\mathbf{v}_1(\mathbf{r}) \cdot \mathbf{v}_2(\mathbf{r}) = c(\mathbf{r}) \tag{B.7}$$

- Cross product: The cross product of two vector field yields to a vector field.

$$\mathbf{v_1}(\mathbf{r}) \times \mathbf{v_2}(\mathbf{r}) = \mathbf{v_3}(\mathbf{r}) \tag{B.8}$$

where $\mathbf{r}$ is a general space(time) vector.

Definition B.4.1 — Nabla operator. Given the unit vectors in 3D Cartesian coordinates $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ the Nabla operator is given as

$$\nabla := \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y + \mathbf{e}_z \partial_z$$

where ∂_i is the short notion for the partial derivative $\frac{\partial}{\partial i}$.

The Nabla in the column vector form

$$\nabla := \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

The three basic vector operators that we can do with this operator are:

- Gradient: This is formulated to the analogy of the scalar multiplication of Nabla and the scalar field:

$$\nabla f = \mathbf{e}_x \partial_x f + \mathbf{e}_y \partial_y f + \mathbf{e}_z \partial_z f \tag{B.9}$$

  and describes the rate and direction of change in the scalar field.
- Divergence: This is formulated to the analogy of the dot product of Nabla and the vector field:

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z \tag{B.10}$$

  and it describes the source or sink of the vector field.

- Curl: This is formulated to the analogy of the cross product of Nabla and the vector field:

$$\nabla \times \mathbf{F} = \mathbf{e}_x(\partial_y F_z - \partial_z F_y) + \mathbf{e}_y(\partial_z F_x - \partial_x F_z) + \mathbf{e}_z(\partial_x F_y - \partial_y F_x) \tag{B.11}$$

  and it describes the rotation of the vector field.

As the outcome of the above one can next introduce the (vector) Laplacian:

Definition B.4.2 — Laplace operator. Given the unit vectors in 3D Cartesian coordinates $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ the Laplacian operator is given as

$$\triangle := \nabla^2 = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2$$

where ∂_i^2 is the short notion for the partial second derivative $\frac{\partial^2}{\partial i^2}$.

B.5 Vector analysis

B.5.1 A "standard" integral

On a "standard" definite integral of a function we can understand the area that is under the function. This is denoted as

$$\int_a^b f(x)dx$$

and understood the following way: we break the x axis into little peaces, so little peaces that we can consider the area as rectangles and then we add these small parts together. Figure B.1. aims to illustrate this.

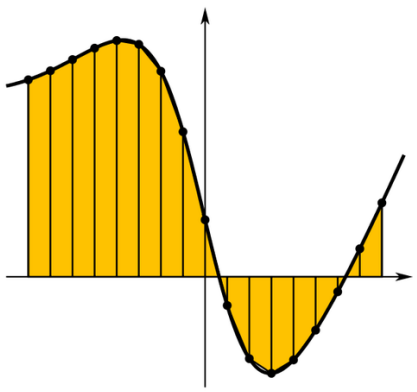


Figure B.1: Ingratiation of a function by dividing into small pieces.

B.5.2 A line integral

A line integral works very similarly. It is denoted as

$$\int_c \mathbf{v}(x,y,z) \cdot d\mathbf{r}$$

where $\mathbf{v}(x,y,z)$ is a vector field, c is the path (a line) that we take. When we break this line into little pieces, each segment will have a direction (the direction of the path) and these small segments with a direction are denoted as $d\mathbf{r}$.

At the midpoint of these little sections we take the value of $\mathbf{v}(x,y,z)$ and dot it together with this line element vector. After this we add all segments up:

$$\sum_{\text{all segments}} \mathbf{v} \cdot d\mathbf{r}$$

When we go with the lengths of these pieces to zero the summation will become an integral and we get back what has been originally denoted as a line integral.

If the path we take ends up where it started we say that the integral is closed and the notion is the following:

$$\oint_c \mathbf{v} \cdot d\mathbf{r}$$

B.5.3 Stokes' theorem

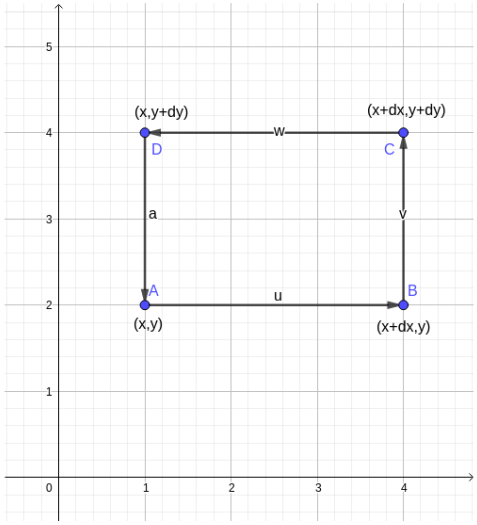


Figure B.2: A simple vector field in a rectange to derive the Stokes theorem.

The closed integral of Figure B.2. is as follows

$$\oint_c \mathbf{v} \cdot d\mathbf{r} = \sum_{i=u,v,w,a} \mathbf{v}_i \cdot d\mathbf{r}_i \tag{B.12}$$

In the followings we go through all of these sections and figure out the values of $\mathbf{v}$ in the midpoint of the sections
in the u section $d\mathbf{r}_i = dx\mathbf{e}_x$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+\frac{dx}{2}, y)$,
in the v section $d\mathbf{r}_i = dy\mathbf{e}_y$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+dx, y+\frac{dy}{2})$,
In the w section $d\mathbf{r}_i = -dx\mathbf{e}_x$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+\frac{dx}{2}, y+dy)$,
in the a section $d\mathbf{r}_i = -dy\mathbf{e}_y$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x, y+\frac{dy}{2})$.

Inserting this information to the Equation B.12:

$$\sum_{i=u,v,w,a} \mathbf{v}_i \cdot d\mathbf{r}_i = dx\mathbf{e}_x \cdot \mathbf{v}\left((x+\frac{dx}{2},y\right) + dy\mathbf{e}_y \cdot \mathbf{v}\left((x+dx,y+\frac{dy}{2}\right) - dx\mathbf{e}_x \cdot \mathbf{v}\left((x+\frac{dx}{2},y+dy\right) - dy\mathbf{e}_y \cdot \mathbf{v}\left((x,y+\frac{dy}{2}\right) \quad \text{(B.13)}$$

Taylor expanding the $\mathbf{v}$ function we get

$$= dx\left(v_x(x,y) + \frac{dx}{2}\frac{\partial v_x}{\partial x}\right) + dy\left(v_y(x,y) + dx\frac{\partial v_y}{\partial x} + dy\frac{\partial v_y}{\partial y}\right) - dx\left(v_x(x,y) + \frac{dx}{2}\frac{\partial v_x}{\partial x} + dy\frac{\partial v_y}{\partial y}\right) - dy\left(v_y(x,y) + \frac{dy}{2}\frac{\partial v_y}{\partial y}\right) \quad \text{(B.14)}$$

where we introduced the notion $v_x = \mathbf{e}_x \cdot \mathbf{v}$ and $v_y = \mathbf{e}_y \cdot \mathbf{v}$.
Luckily, many things cancel out, we we are left over is

$$= dxdy\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \quad \text{(B.15)}$$

This happens to be what we see in Equation B.11. Introducing the notion of $dA = dxdy$

$$= dA\mathbf{e}_z \cdot (\nabla \times \mathbf{v}) \quad \text{(B.16)}$$

Introducing the notion of $d\mathbf{A} = dA\mathbf{e}_z$ we can reach the topic of an **area/surface integral**. We follow the same idea as before, if we have a surface we divide it into small dA segments. The orientation of the boundary of a surface tells us the direction of $d\mathbf{A}$ following the right hand rule.

Summarizing our results

$$\oint_c \mathbf{v} \cdot d\mathbf{r} = \int_A (\nabla \times \mathbf{v})\, d\mathbf{A} \quad \text{(B.17)}$$

At this moment this is true for small area. If we have a bigger area then let's divide it into small pieces where the above will be true! The theorem then holds for any A and it boundary (notion is $c = \partial A$). Which leads us to the theorem.

Theorem B.5.1 — Stokes' theorem. A closed integral of a vector field over a boundary of a surface is the same as a surface integral of the curl of the vector field:

$$\oint_{\partial A} \mathbf{v} \cdot d\mathbf{r} = \int_A (\nabla \times \mathbf{v})\, d\mathbf{A}$$

B.5.4 Gauss' theorem

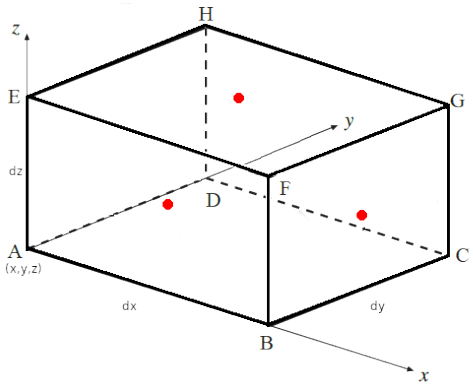


Figure B.3: A simple vector field in a box to derive the Gauss' theorem.

The closed surface integral of a vector field in a box as in Figure B.3 has the the form of

$$\oint_A \mathbf{v} \cdot d\mathbf{A} = \sum_{i=6 sides} \mathbf{v}_i \cdot d\mathbf{A}_i \tag{B.18}$$

In the followings we go through all of these sections and figure out the values of $\mathbf{v}$ in the midpoints (denoted with red on the figure)
in the bottom side $d\mathbf{A}_i = -dxdy\mathbf{e}_z$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+\frac{dx}{2}, y+\frac{dy}{2}, z)$,
in the top side $d\mathbf{A}_i = dxdy\mathbf{e}_z$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+\frac{dx}{2}, y+\frac{dy}{2}, z+dz)$,
in the left side $d\mathbf{A}_i = -dydz\mathbf{e}_x$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x, y+\frac{dy}{2}, z+\frac{dz}{2})$,
in the right side $d\mathbf{A}_i = dydz\mathbf{e}_x$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+dx, y+\frac{dy}{2}, z++\frac{dz}{2})$,
in the front side $d\mathbf{A}_i = -dxdz\mathbf{e}_y$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+\frac{dx}{2}, y, z+\frac{dz}{2})$,
in the back side $d\mathbf{A}_i = dxdz\mathbf{e}_y$ and $\mathbf{v}_i$ is considered at the values $\mathbf{v}(x+\frac{dx}{2}, y+dy, +\frac{dz}{2})$

Inserting this information to the Equation B.18:

$$\begin{aligned}\sum_{i=6 sides} \mathbf{v}_i \cdot d\mathbf{A}_i = &-\mathbf{v}\left(x+\frac{dx}{2}, y+\frac{dy}{2}, z\right) \cdot dxdy\mathbf{e}_z + \mathbf{v}\left(x+\frac{dx}{2}, y+\frac{dy}{2}, z+dz\right) \cdot dxdy\mathbf{e}_z \\ &-\mathbf{v}\left(x, y+\frac{dy}{2}, z+\frac{dz}{2}\right) \cdot dydz\mathbf{e}_x + \mathbf{v}\left(x+dx, y+\frac{dy}{2}, z++\frac{dz}{2}\right) \cdot dydz\mathbf{e}_x \\ &-\mathbf{v}\left(x+\frac{dx}{2}, y, z+\frac{dz}{2}\right) \cdot dxdz\mathbf{e}_y + \mathbf{v}\left(x+\frac{dx}{2}, y+dy, +\frac{dz}{2}\right) \cdot dxdz\mathbf{e}_y\end{aligned} \tag{B.19}$$

Taylor expanding the $\mathbf{v}$ function for the top and bottom part, we get

$$\begin{aligned}\sum_{i=2 sides} \mathbf{v}_i \cdot d\mathbf{A}_i = &\left[-v_z(x,y,z)dxdy - \frac{dx}{2}dxdy\frac{\partial v_z}{\partial x} - \frac{dy}{2}dydx\frac{\partial v_z}{\partial y}\right] + \\ &\left[v_z(x,y,z)dxdy + \frac{dx}{2}dxdy\frac{\partial v_z}{\partial x} + \frac{dy}{2}dydx\frac{\partial v_z}{\partial y} + dxdydz\frac{\partial v_z}{\partial z}\right]\end{aligned} \tag{B.20}$$

from which only the $dxdydz\frac{\partial v_z}{\partial z}$ term survives. After repeating this for the right/front and left/back we get that

$$\sum_{i=6\,sides} \mathbf{v}_i \cdot d\mathbf{A}_i = dxdydz\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] \tag{B.21}$$

Introducing the notion $dV = dxdydz$ and realizing the definition of the divergence in the bracket we can write this as

$$= (\nabla \cdot \mathbf{v})dV \tag{B.22}$$

Summarizing our results

$$\oint_A \mathbf{v} \cdot d\mathbf{A} = \int_V (\nabla \cdot \mathbf{v})\, dV \tag{B.23}$$

Similarly to Stokes' theorem if we have a bigger volume we can always divide it up to smaller volumes where this will be true. The theorem then holds for any V and it boundary surface (notion $A = \partial V$). Which leads us to the theorem.

Theorem B.5.2 — Gauss' theorem. A closed surface integral of a vector field is the same as a volume integral of the divergence of the vector field:

$$\oint_{\partial V} \mathbf{v} \cdot d\mathbf{A} = \int_V (\nabla \cdot \mathbf{v})\, dV$$

B.6 Coordinate systems

B.6.1 Polar coordinates

$$\begin{aligned} x &= rcos(\phi) \\ y &= rsin(\phi) \end{aligned} \tag{B.24}$$

where r is the radius and ϕ is an angle going from $[0, 2\pi]$.

B.6.2 Elliptic coordinates

$$\begin{aligned} x &= acos(\phi)cosh(b) \\ y &= asin(\phi)sinh(b) \end{aligned} \tag{B.25}$$

where a is the distance between the focii, ϕ is an angle going from $[0, 2\pi]$, and b is real number.

B.6.3 Cylindrical coordinates

$$\begin{aligned} x &= rcos(\phi) \\ y &= rsin(\phi) \\ z &= z \end{aligned} \tag{B.26}$$

where r is the radius, ϕ is an angle going from $[0, 2\pi]$ and z is the third coordinate.

B.6.4 Spherical coordinates

$$\begin{aligned} x &= rcos(\phi)sin(\theta) \\ y &= rsin(\phi)sin(\theta) \\ z &= rcos(\theta) \end{aligned} \tag{B.27}$$

where r is the radius, ϕ is an angle going from $[0, 2\pi]$ and θ is an angle going from $[0, \pi]$

B.6.5 Jacobian matrix and determinant

In vector calculus, the Jacobian matrix is the matrix of all first-order partial derivatives of a vector-valued function. When the matrix is a square matrix, both the matrix and its determinant are referred to as the Jacobian in literature.

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{B.28}$$

The way to calculate its determinant depends on the size.
In the case of a 2 x 2 matrix the determinant may be defined as:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{B.29}$$

Similarly, for a 3 x 3 matrix "A", its determinant is:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \tag{B.30}$$

(B.31)

This leads to

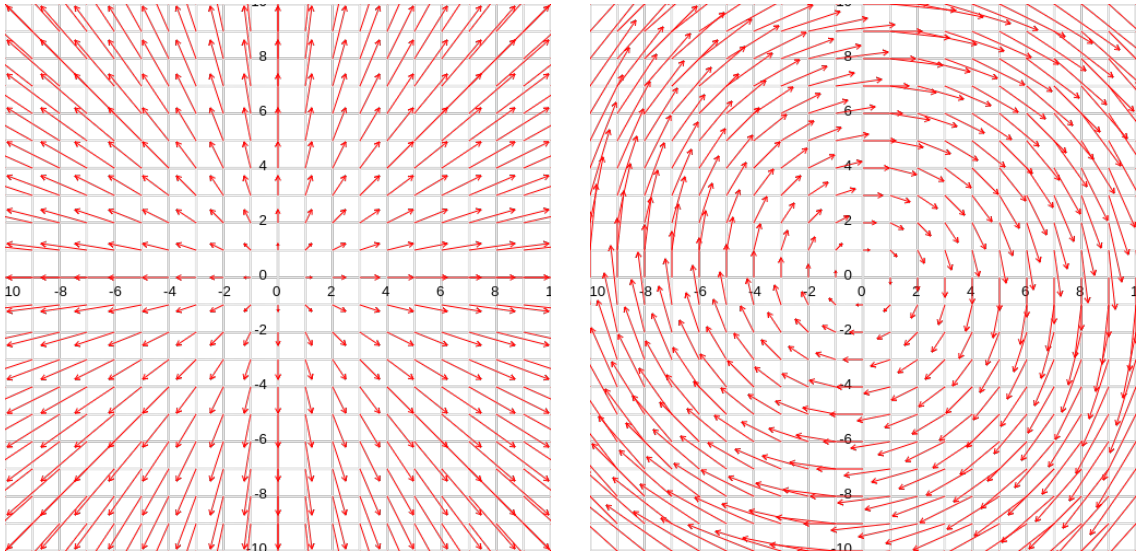
$$|A| = aei + bfg + cdh - ceg - bdi - afh \tag{B.32}$$

B.7 Exercises

Exercise B.1 Calculate the divergence and curl of the following fields: (a) $\mathbf{F}(\mathbf{r}) = x\mathbf{e}_x + y\mathbf{e}_y$ (b) $\mathbf{F}(\mathbf{r}) = y\mathbf{e}_x - x\mathbf{e}_y$. See the plots in Figure B.4. Which one circulates? ■

Exercise B.2 Calculate the gradient, divergence and curl of the following fields: (a) $f(\mathbf{r}) = x^2y - z$ and (b) $\mathbf{F}(\mathbf{r}) = x\mathbf{e}_x - xy\mathbf{e}_y + z^2\mathbf{e}_z$. ■

Exercise B.3 Calculate the Jacobian determinant for the polar and spherical coordinates. ■



(a) (b)

Figure B.4: Plotting the vector fields mentioned in Exercise B.1